Latin squares without proper subsquares

Jack Allsop

Monash University

Joint work with Ian Wanless

Latin Squares

Definition

Let n be a positive integer. A *Latin square* of order n is an $n \times n$ matrix of n symbols such that each symbol occurs exactly once in each row and column.

1	2	3	4	5
2	4	1	5	3
3	5	4	2	1
4	1	5	3	2
5	3	2	1	4

Subsquares

Definition

Let L be a Latin square of order n. A *subsquare* of L is a submatrix of L which is itself a Latin square. A subsquare of L of order k is *proper* if $1 < k < n$.

6	1	5	4	3	2
4	5	1	6	2	3
2	4	6	3	5	1
1	3	2	5	6	4
3	6	4	2	1	5
5	2	3	1	4	6

N_∞ Latin squares

Definition

A Latin square is called N_∞ if it contains no proper subsquares.

2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4
1	2	3	4	5

N_∞ Latin squares

Definition
A Latin square is called N_∞ if it contains no proper subsquares.

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The Cayley table of a cyclic group of prime order is N_∞.

N_∞ Latin squares

n	Species of Latin squares	Species of N_∞ Latin squares
1	1	1
2	1	1
3	1	1
4	2	0
5	2	1
6	12	0
7	147	2
8	283657	3
9	19270853541	1589

N_2 Latin squares are rare. Kwan, Sah, Sawhney and Simkin showed that the probability of a random Latin square of order n being N_2 is $\exp(-\Omega(n^2))$.

N_∞ Latin squares

Theorem

There exists an N_∞ Latin square of order n for all n not of the form 2^a3^b with $a \geqslant 1$ and $b \geqslant 0$.

- It is conjectured that there exists an N_∞ square of order n for all sufficiently large n (Hilton 1974).
- There exists an N_∞ square of order pq whenever p and q are distinct primes with $pq \neq 6$ (Heinrich 1980).
- There exists an N_∞ square of all orders not of the form 2^a3^b with $a \geqslant 0$ and $b \geqslant 0$ (Andersen and Mendelsohn 1982).
- There exists an N_∞ square of order $3m$ for all odd integers m (Maenhaut, Wanless, and Webb 2007).

N_∞ Latin squares

We construct an N_∞ of order n for each n of the form $2^a 3^b \notin \{4, 6\}$ with $a \geqslant 1$ and $b \geqslant 0$.

Theorem
There exists an N_∞ Latin square of order n for all $n \notin \{4, 6\}$.

Direct products

Let L be a Latin square of order n and let M be a Latin square of order m. The direct product of L and M, denoted by $L \times M$, is a Latin square of order nm whose row indices, column indices and symbols are in the set $[n] \times [m]$. It is defined by

$$(L \times M)_{(i,j),(k,\ell)} = (L_{i,k}, M_{j,\ell}).$$

Consider the following two Latin squares.

$$L = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}, \qquad M = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}.$$

Direct products

The following is $L \times M$ where we order the rows and columns by the first coordinate, and use the second to break ties.

(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)
(1, 2)	(1, 3)	(1, 1)	(2, 2)	(2, 3)	(2, 1)
(1, 3)	(1, 1)	(1, 2)	(2, 3)	(2, 1)	(2, 2)
(2, 1)	(2, 2)	(2, 3)	(1, 1)	(1, 2)	(1, 3)
(2, 2)	(2, 3)	(2, 1)	(1, 2)	(1, 3)	(1, 1)
(2, 3)	(2, 1)	(2, 2)	(1, 3)	(1, 1)	(1, 2)

Direct products

The following is $L \times M$ where we order the rows and columns by the second coordinate, and use the first to break ties.

(1, 1)	(2, 1)	(1, 2)	(2, 2)	(1, 3)	(2, 3)
(2, 1)	(1, 1)	(2, 2)	(1, 2)	(2, 3)	(1, 3)
(1, 2)	(2, 2)	(1, 3)	(2, 3)	(1, 1)	(2, 1)
(2, 2)	(1, 2)	(2, 3)	(1, 3)	(2, 1)	(1, 1)
(1, 3)	(2, 3)	(1, 1)	(2, 1)	(1, 2)	(2, 2)
(2, 3)	(1, 3)	(2, 1)	(1, 1)	(2, 2)	(1, 2)

Corrupted products

- Let A be an N_∞ square of order n.
- Let B be a Latin square isotopic to A, such that $A[i,j] = B[i,j]$ if and only if $i = j = 1$.
- Let M be an N_∞ square of order m.
- Let $s \in [m-1]$.

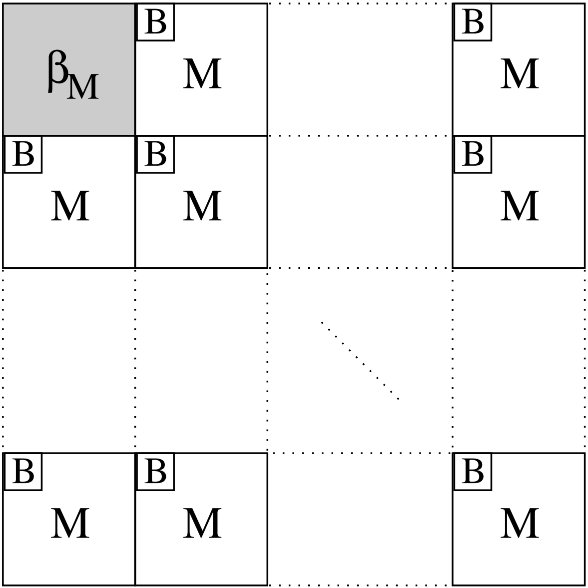
Corrupted products

The corrupted product of (A, B) and M with shift s, denoted by $P = (A, B) *_s M$, is a Latin square of order nm whose row indices, column indices, and symbols are in the set $[n] \times [m]$. It is defined by,

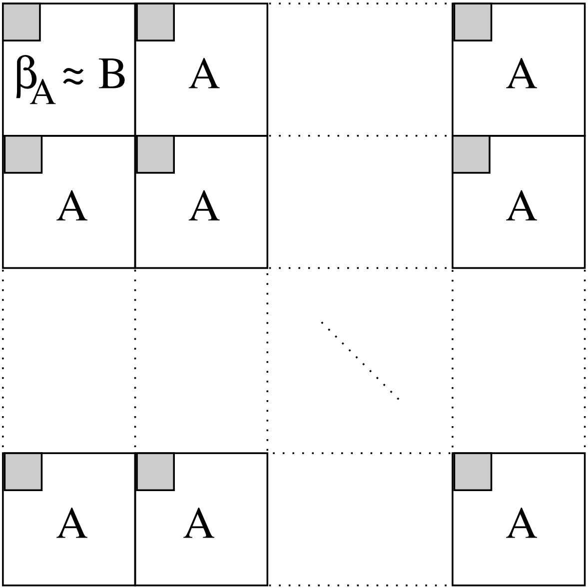
$$P[(i,j),(k,l)] = \begin{cases} (A[i,k], M[j,l] + s) & \text{if } i = k = 1, \\ (B[i,k], M[j,l]) & \text{if } j = l = 1 \text{ and } (i,k) \neq (1,1), \\ (A[i,k], M[j,l]) & \text{otherwise.} \end{cases}$$

This was introduced by Wanless.

Corrupted products



Corrupted products



Corrupted products

Theorem (Wanless 2001)

The corrupted product $(A, B) *_s M$ *has only one proper subsquare (provided some mild conditions on* (A, B)*,* M *and* s *hold).*

Cycle switches

1	2	3	4	5
2	4	1	5	3
3	5	4	2	1
4	1	5	3	2
5	3	2	1	4

$\longrightarrow$

1	2	3	4	5
5	4	2	1	3
3	5	4	2	1
4	1	5	3	2
2	3	1	5	4

Constructing N_∞ squares

We used a recursive construction to build N_∞ Latin squares for all orders of the form $2^a 3^b$. Our method was as follows:

- Find a pair (A_8, B_8) of Latin squares of order 8 and a pair (A_9, B_9) of Latin squares of order 9 which satisfies certain properties.
- Given an N_∞ square M of order m which satisfies some nice properties, construct corrupted products $(A_8, B_8) *_1 M$ of order $8m$ and $(A_9, B_9) *_1 M$ of order $9m$, both of which have a unique subsquare.
- Switch the corrupted product on a row cycle of length three which hits the unique subsquare exactly once, in such a way as to not introduce any new subsquares.

Conclusion

- We resolved the existence problem for N_∞ Latin squares and N_∞ Latin hypercubes.
- It is likely that a similar approach would work to construct N_∞ squares of other orders.

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