Groups of Lie Type Acting on Generalised Quadrangles

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- E_6, E_7, E_8, F_4, G_2

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- Rank 3 and above classified* by Weiss and Tits
- Rank 2 remains unclassified

Generalised Polygons

Generalised n-gon

- Two points lie in at most one line, two lines intersect in at most one point
- No k-gons for $k \in \{3, \dots, n-1\}$
- Any two elements (points or lines) is contained in an n-gon

- k-gon: sequence of points $a_0, \dots, a_{k-1}$ where a_i and a_{i+1} lie in a common line ($+$ is mod k)

Generalised Polygons

- **Flag** - point-line incident pair
- **Order** (s, t): every line contains $s + 1$ points and every point lies on $t + 1$ lines
- **Thick**: $s, t \geqslant 2$
- Higman and Feit (1964): thick implies $n \in \{2, 3, 4, 6, 8\}$

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- $n = 6$: two infinite families of generalised hexagons (and their duals)
- $n = 8$: one infinite family of generalised octagon (and its dual)

Generalised Quadrangle

- Two points lie in at most one line, two lines intersect in at most one point
- Given a line L and a point x not on L, there is a unique point y on L such that x and y are on a line

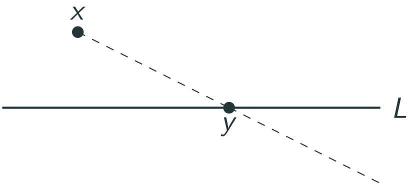

Figure 1: Second GQ Axiom

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- Automorphism group is $\mathrm{P\Gamma Sp}_4(q)$

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- Other examples: **flock quadrangles**
- Can we classify them?

Transitivity and Primitivity

Let G act on a set Ω.

- Transitive: for every $\alpha, \beta \in \Omega$, there is a $g \in G$ such that $\alpha^g = \beta$
- Primitive: G preserves only the trivial partitions of Ω
  - $\{\Omega\}$
  - $\{\{\alpha\} : \alpha \in \Omega\}$

- Point-primitivity
- Line-primitivity
- Flag-transitivity (recall: flag is a point-line incident pair)

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Other Symmetry Conditions

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Bamberg, Li, Swartz (2018)

Let G act **antiflag-transitively** on a generalised quadrangle.
Then that generalised quadrangle is $\mathrm{GQ}(3,5)$ or $\mathrm{GQ}(5,3)$.

Hypothesis Let G act **point-primitively**, **line-primitively** and **flag-transitively** on a generalised hexagon or octagon

Schneider, Van Maldeghem (2008)

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Conclusion Then G is almost simple of Lie type

Almost simple $S \leq G \leq \mathrm{Aut}(S)$ where S is a non-abelian simple group

Bamberg, Glasby, Popiel, Praeger, Schneider (2017)

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Conclusion Then G_α is not the stabiliser in G of a subspace of the natural module $V = (\mathbb{F}_q)^n$.

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Conclusion Then the generalised polygon is the classical generalised octagon or its dual.

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Conclusion Then G is almost simple of Lie type.

Bamberg, Evans (2021)

- No sporadic almost simple group can act **primitively** on **points** of any generalised quadrangle.

Feng, Lu (2023)

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Conclusion Then $q = 9$ and Γ is the symplectic quadrangle $W(2)$.

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- Maximal subgroups of G:
  - $E_q.E_q.C_{q-1}$, where E_q is elementary abelian, C_{q-1} is cyclic
  - $D_{2(q-1)}$
  - $C_{q\pm\sqrt{2q}+1} : C_4$
  - ${}^2B_2(q_0)$, where $q_0^r = q$, $q_0 > 2$ and r is prime

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- Go through cases

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- $s + 1$ must have a "small" 2-part
- Conjecture: there are no solutions for s, t satisfying the conditions above
- Suggested by numerical evidence

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  - $C_{q \pm \sqrt{3q}+1} : C_6$
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