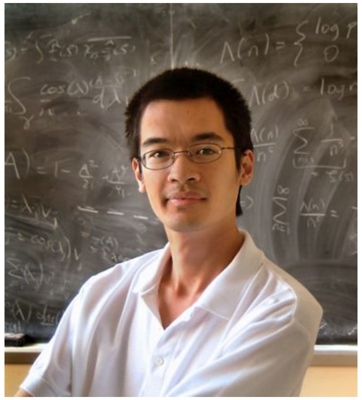
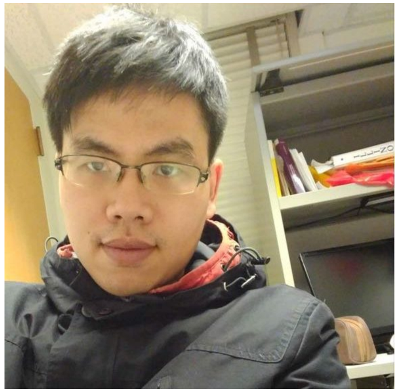
Point-box incidences and logarithmic density of semilinear graphs

Abdul Basit
Monash University

Joint work with

Artem Chernikov, Sergei Starchenko, Terence Tao, and Chieu-Minh Tran

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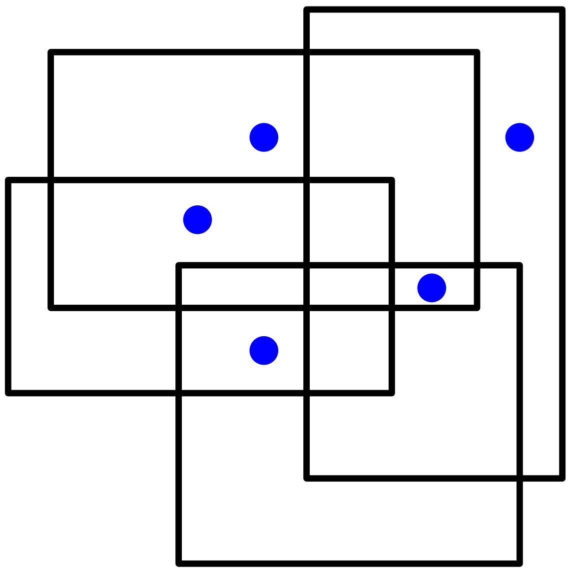

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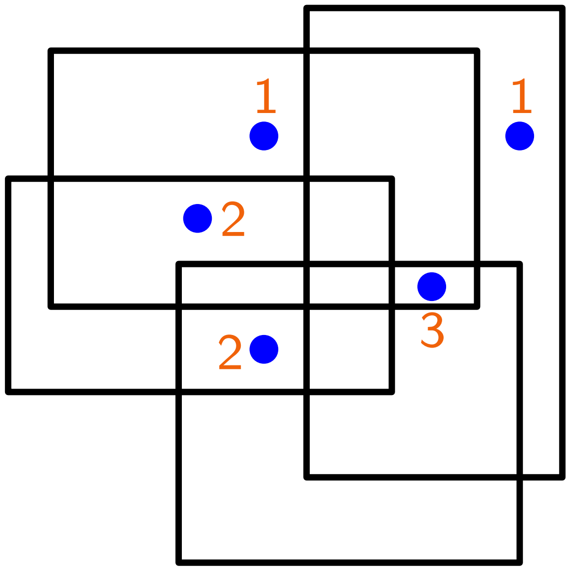

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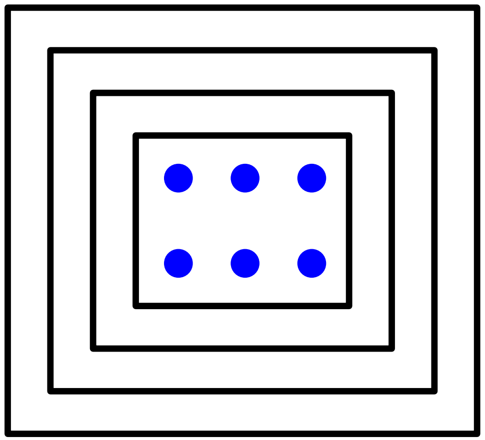

Without the assumption that no k boxes have k points in common, there could be $n_1 \cdot n_2$ incidences.

Zarankiewicz's problem

A question in extremal graph theory:
For $k \in \mathbb{N}$, let $K_{k,k}$ be the complete bipartite graph with k vertices in each block.

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If $G = (V_1, V_2; E)$ with $|V_1| + |V_2| = n$ is $K_{k,k}$-free, then $|E| \leq O_k\left(n^{2-1/k}\right)$.

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Known bounds:
$k = 2$: Incidence graph of a finite projective plane (Klein '38?)
$k = 3$: Point-sphere incidence graphs in $\mathbb{F}_p^3$ for $p > 3$ (Brown '66).
Projective norm graphs (Alon-Rónyai-Szabó '99).

Open for $k \geq 4$

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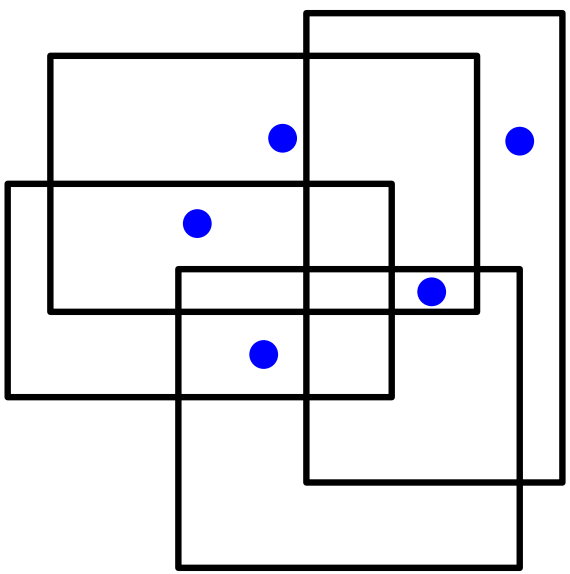

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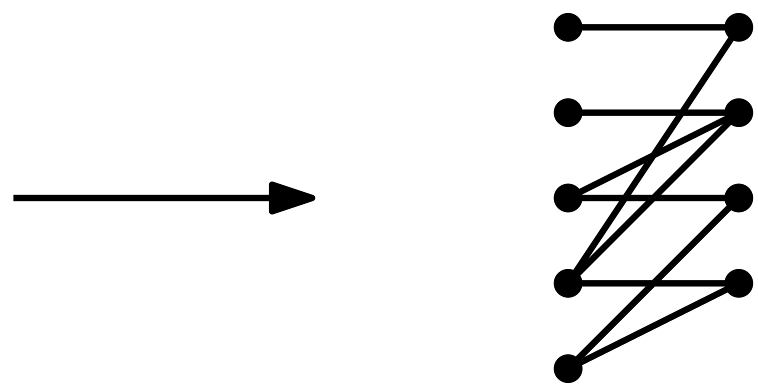
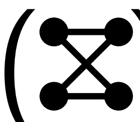


No $K_{2,2}$ ()

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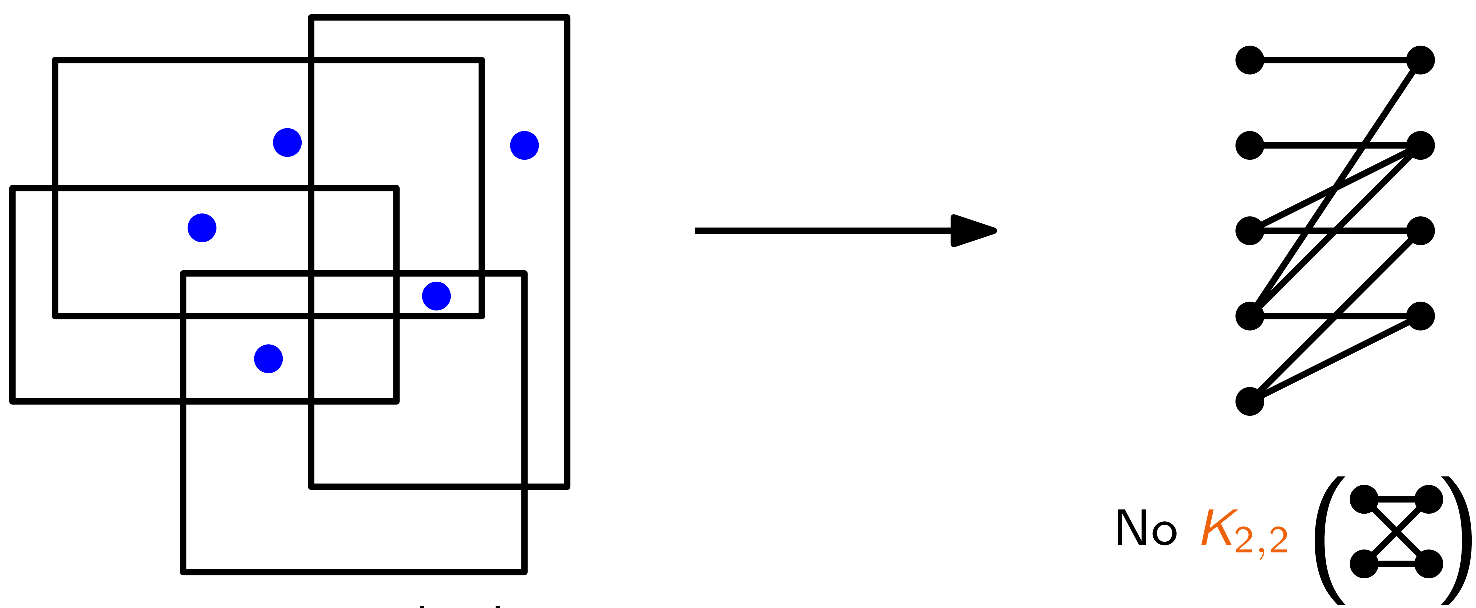

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If no k rectangles have k points in common, then G is $K_{k,k}$-free. So by the KST Theorem, the number of incidences is $O_k\left(n^{2-1/k}\right)$.

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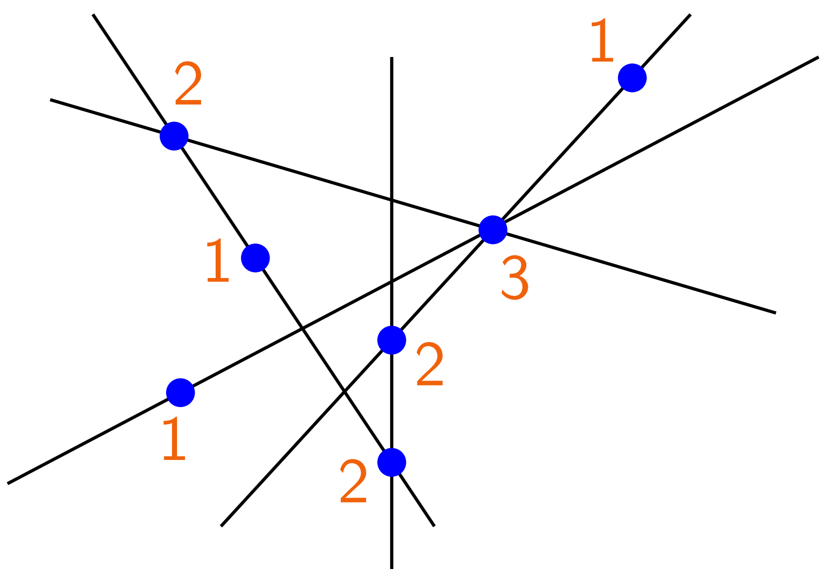

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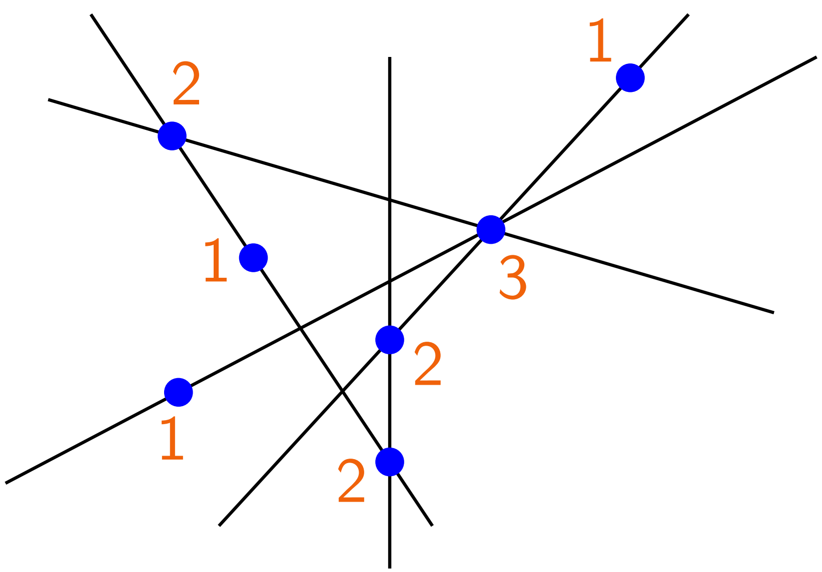

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This is optimal, i.e., there exist configurations with $\Omega\left(n^{4/3}\right)$ incidences.

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- Erdős-Szemerédi sum-product conjecture
- Erdős distinct distance and unit distance conjectures
- Harmonic Analysis, Number Theory, Model Theory, Computer Science, and more

Zarankiewicz's problem for semialgebraic graphs

A graph $G = (V_1, V_2; E)$ is semialgebraic if $V_1 \subset \mathbb{R}^{d_1}$, $V_2 \subset \mathbb{R}^{d_2}$ and there exists a system of t *polynomial inequalities* $\varphi(x, y)$ of degree at most D such that $E = \{(a, b) \in V_1 \times V_2 : \varphi(a, b)\}$.

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Theorem (Fox-Pach-Sheffer-Suk-Zahl '12):
Let $G = (V_1, V_2; E)$ be a semialgebraic graph of constant complexity s, with $|V_1| + |V_2| = n$. If G is $K_{k,k}$-free, then $|E| = O_{k,s}\left(n^{2-c}\right)$, where $0 < c < 1$ depends only on d_1 and d_2.

Common generalization of many geometric incidence results.

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Why we should expect better bounds:

In the point-line incidence graph, E is defined by the inner product, using *addition and multiplication*.

In the point-rectangle incidence graph, E is defined using only *ordering*.

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Theorem 1 (B.-Chernikov-Starchenko-Tao-Tran '20):

(i) Given n_1 points and n_2 axis-parallel rectangles in $\mathbb{R}^2$ (with $n = n_1 + n_2$), if no k rectangles have k points in common, the number of incidences is $O_k\left(n \log^4 n\right)$.

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Har-Peled and Chan '22:
For any set of points and rectangles, the number of incidences is $O_k\left(n \frac{\log n}{\log \log n}\right)$

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Theorem 2 (B.-Chernikov-Starchenko-Tao-Tran '20):
Let $G = (V_1, V_2; E)$ be a semilinear graph with $|V_1| + |V_2| = n$. If G is $K_{k,k}$-free, then $|E| = O_{k,\varphi}\left(n \log^s n\right)$.

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More generally:
Any ordered division ring instead of $\mathbb{R}$.
Functions that are coordinate-wise monotone.

A function $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ is coordinate-wise monotone if for any $a, a' \in V_1 \in \mathbb{R}^{d_1}$ and $b, b' \in \mathbb{R}^{d_2}$, we have

- $f(a, b) \leq f(a, b') \iff f(a', b) \leq f(a', b')$
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Proof Idea:

Induction on number of linear equations s.

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Use the order structure of $\mathbb{R}$ to split up the graph and control incidences.

Suppose L is one of the defining inequalities. Assume L has the form $L_1(x) < L_2(y)$ with $L_1 : V_1 \to \mathbb{R}$ and $L_2 : V_2 \to \mathbb{R}$.

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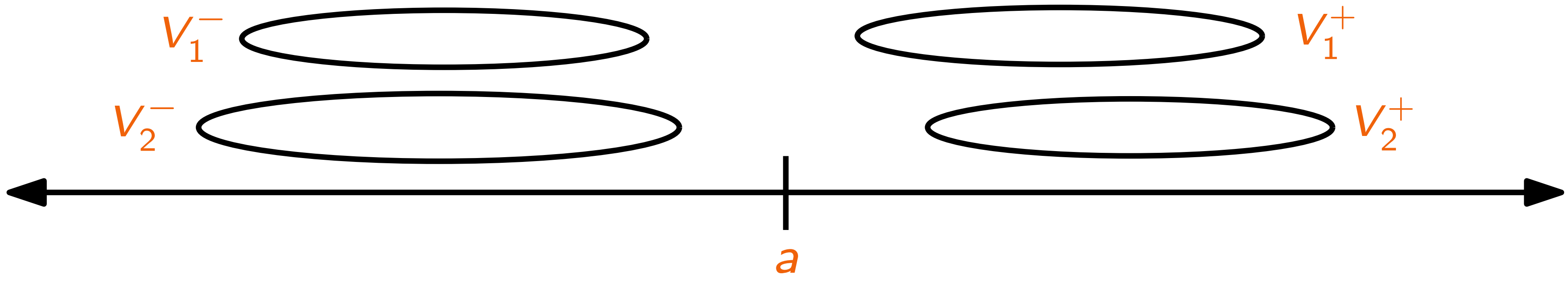

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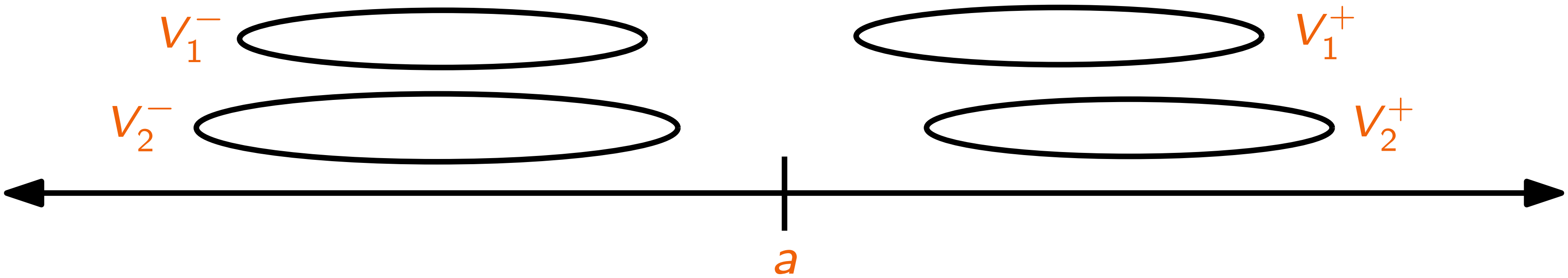

- $|E \cap (V_1^- \times V_2^-)| \leq f_s\left(\lfloor \frac{n}{2} \rfloor\right)$
- $|E \cap (V_1^+ \times V_2^+)| \leq f_s\left(\lfloor \frac{n}{2} \rfloor\right)$

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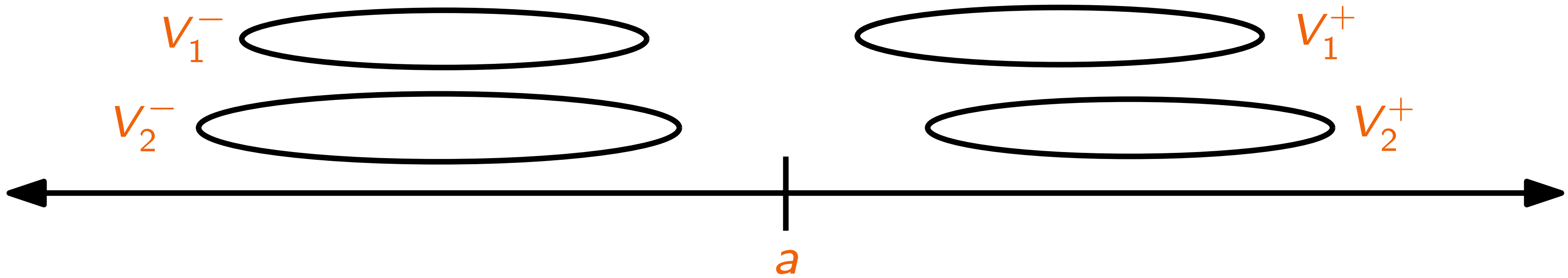
Let $a \in \mathbb{R}$ be a point that bisects $L_1(V_1) \cup L_2(V_2)$.



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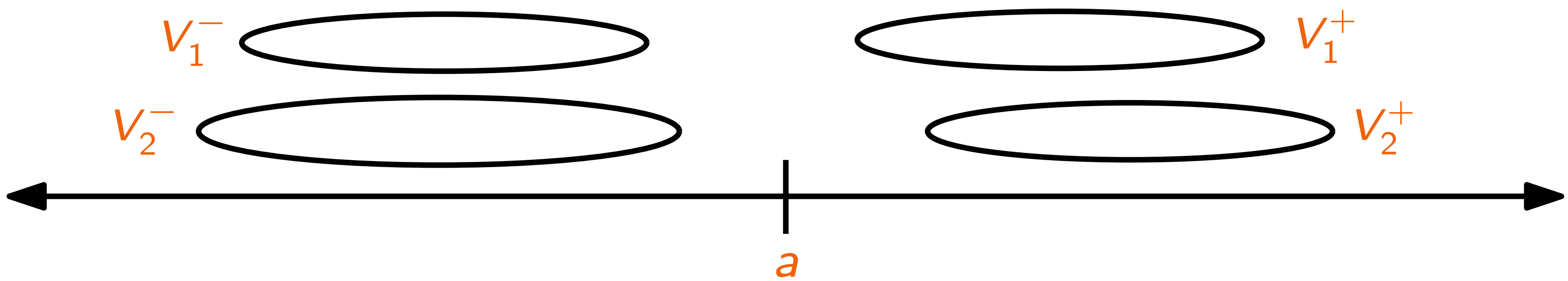

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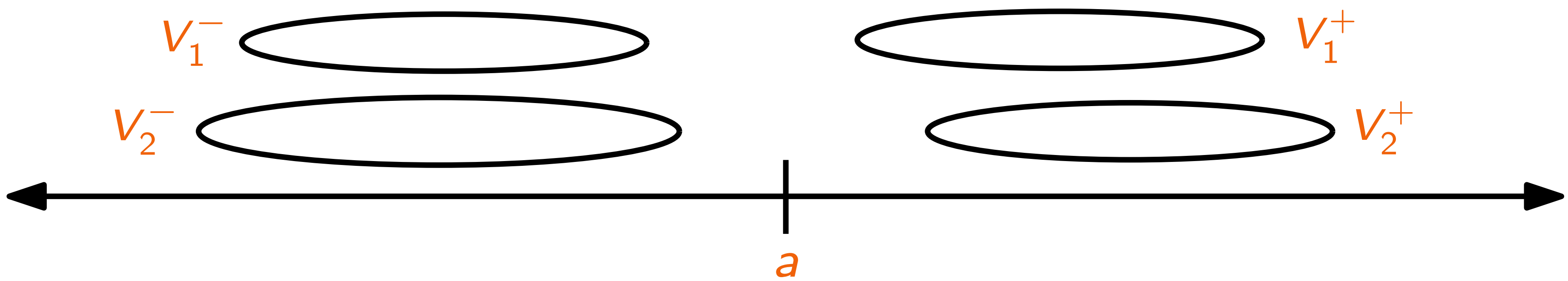

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Extensions to Hypergraphs

Theorem (Erdős '64):
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a r-partite r-uniform hypergraph with $|V_1| + \cdots + |V_r| = n$. If H is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k}\left(n^{r - \frac{1}{k^{r-1}}}\right)$.

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Theorem 3 (B.-Chernikov-Starchenko-Tao-Tran '20):
Let $H = (V_1, V_2, \ldots, V_r, E)$ be a semilinear hypergraph with $|V_1| + \cdots + |V_r| = n$. If H is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k,\varphi}\left(n^{r-1} \log^c n\right)$ where c depends only on r and the number of defining inequalities.

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Point-polytope incidences:

Given n_1 points and n_2 polytopes in $\mathbb{R}^d$ cut out from half-spaces with normal vectors in a fixed finite set, such that the incidence graph does not contain $K_{k,k}$, the number of incidences is $O\left(n^{1+\varepsilon}\right)$ for any $\varepsilon > 0$.

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Erdős Unit Distance Conjecture: In the ℓ_2 norm, the number of unit distances determined by any set of n points in $\mathbb{R}^2$ is $O(n^{1+\varepsilon})$.

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In fact, given an infinite structure, there is at least one structure per infinite cardinality with the same theory.

E.g., any *real closed field* has the same theory as $(\mathbb{R}, <, +, \times)$.

- real algebraic numbers
- hyperreal numbers
- computable numbers

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Example: Semialgebraic graphs are precisely the definable graphs in $(\mathbb{R}, <, +, \times)$.

Example: Semilinear graphs are definable graphs in o-minimal modular structures.

Theorem 3, along with results in model theory, gives a combinatorial characterisation of modularity for o-minimal structures.

Future work

- Ramsey properties (some results have been obtained by Tomon-Jin).
- Quantitive improvements on the regularity lemma.
- Find more general families that satisfy a linear bound for Zarankiewicz's problem (e.g., definable graphs in abelian groups).

Thank you!