List Colouring and Maximal Local Edge Connectivity

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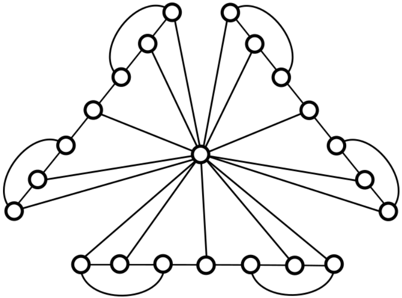
Colouring

A (proper) k-colouring of a graph assigns one of k colours to each vertex of a graph such that no two adjacent vertices have the same colour.



A graph is k-colourable if it permits a k-colouring. Is this graph 3-colourable?

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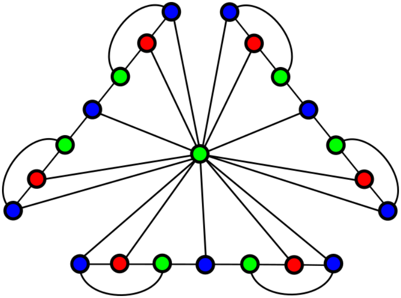


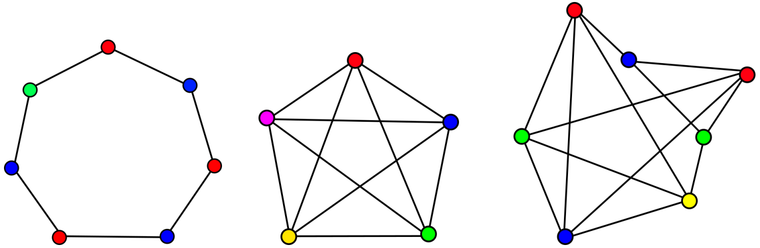
A graph is k-colourable if it permits a k-colouring. Is this graph 3-colourable?

Yes

Brooks' Theorem

Theorem (Brooks 1941)

Let G be a connected graph with maximum degree k. Then G is k-colourable if and only if G is not a complete graph or an odd cycle.



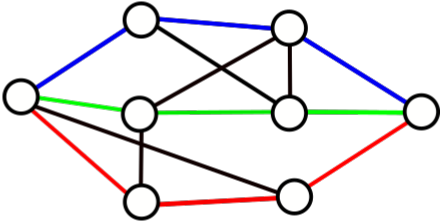
(a) An odd cycle with maximum degree 2 that is not 2-colourable

(b) A complete graph with maximum degree 4 that is not 4-colourable

(c) A graph with maximum degree 4 that is 4-colourable

Connectivity

A graph is *k*-connected if there are k internally vertex-disjoint paths between any two vertices in the graph.



A block is a maximal subgraph that is either 2-connected or isomorphic to K_2.

Maximal Local Connectivity

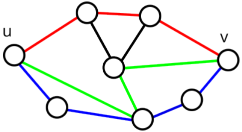
A graph has maximal local connectivity k if there are at most k internally vertex disjoint paths between any two vertices in the graph.

Proposition (Aboulker, Brettell, Havet, Marx, Trotignon, 2018)

The problem of deciding if a 2-connected graph with maximal local connectivity 3 is 3-colourable is NP-complete.

Maximal Local Edge Connectivity

The local edge connectivity between u and v is the maximum number of edge-disjoint paths between u and v.



A graph has maximal local edge connectivity k if the local edge connectivity between any two vertices is at most k.

If a graph has maximum degree k then it has maximal local edge connectivity k. If a graph has maximal local edge connectivity k then it has maximal local connectivity k.

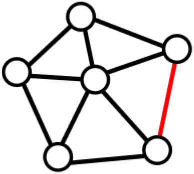
Maximal Local Edge Connectivity

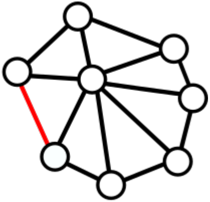
Theorem (Stiebitz and Toft, 2018)

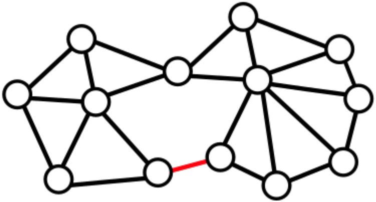
Let G be a graph with maximal local edge connectivity k, $k \geq 3$. Then G is k-colourable if and only if it has no block in $\mathcal{H}_k$.

The class $\mathcal{H}_3$ is the class of Hajós joins of odd wheels. The class $\mathcal{H}_k$ $(k > 3)$ is the class of Hajós joins of K_k graphs.

Hajós Joins

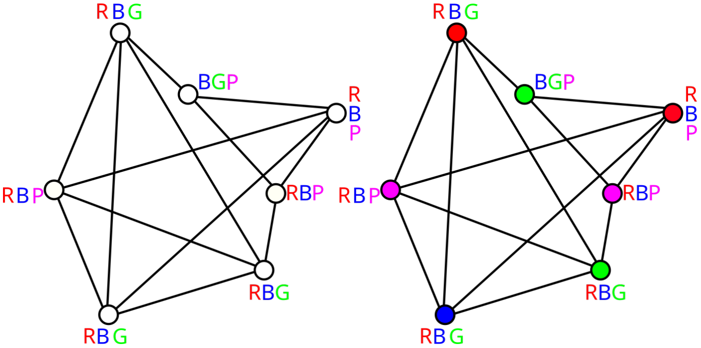






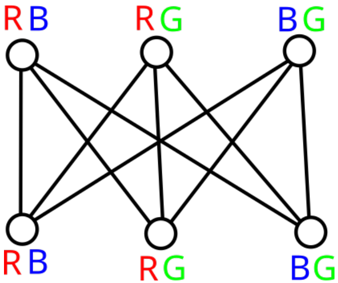
List Colouring

A list assignment, ϕ, of a graph, assigns a list of colours to each vertex. A ϕ-colouring assigns a colour to each vertex, v, from $\phi(v)$ such that no adjacent vertices are assigned the same colour.



List Colouring

A k-list assignment is a list assignment for which $|\phi(v)| = k$ for all v. A graph, G, is k-choosable if there exists a ϕ-colouring for every k-list assignment ϕ of G.



Theorem (Erdős, Rubin and Taylor, 1979)

Let G be a connected graph with maximum degree k. Then G is k-choosable if and only if G is not a complete graph or an odd cycle.

List Colouring for Maximal Local Edge Connectivity

Lemma (B and Brettell, 2023+)

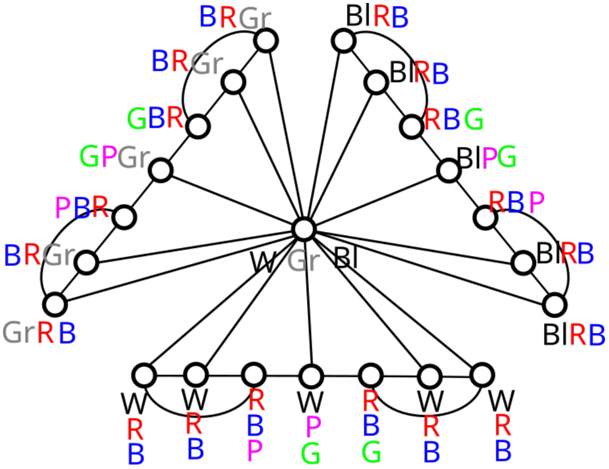
Let G be a k-connected graph with maximal local edge connectivity k. Then G is k-choosable if and only if it is k-colourable ($k \geq 3$).

So can we extend this conjecture to when G is not k-connected?

Naive Conjecture

Let G be a graph with maximal local edge connectivity k, $k \geq 3$. Then G is k-choosable if and only if it is k-colourable.

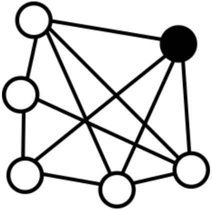
List Colouring for Maximal Local Edge Connectivity

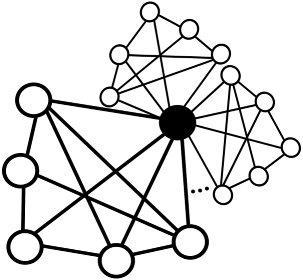


This graph is 3-colourable as shown previously but not ϕ-colourable for the 3-list assignment, ϕ, pictured above.

Restricted Vertices

When a graph G is ϕ-colourable, we say a vertex, v, is ϕ-restricted if there is a colour $c \in \phi(v)$ such that in no ϕ-colouring v is assigned c.





Unrestricted Graphs

We say a graph, G, is ϕ-unrestricted if it is ϕ-colourable and no vertex of G is ϕ-restricted. We say G is k-unresticted if G is ϕ-unrestricted for any k-list assignment, ϕ.

Finding Restricted Vertices

Theorem (B and Brettell, 2023+)

Let G be a k-connected graph with maximal local edge connectivity k. If G is not a complete graph and G is not an odd wheel then G is k-unrestricted.

k=3

In the case when $k = 3$ things break down quite nicely.

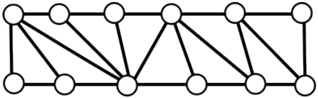
If G is not 2-connected then we decompose into blocks.

If G is 3-connected then we are done by the last lemma.

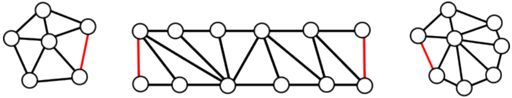
It remains to deal with graphs that are 2-connected but not 3-connected.

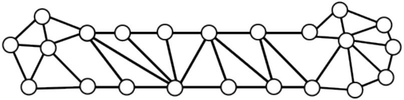
Graphs that are 2-connected but not 3-connected

Helix



Helix join





Graphs that are 2-connected but not 3-connected

Lemma (B and Brettell, 2023+)

Let G be a graph that is 2-connected but not 3-connected. If G has maximal local edge connectivity 3 then G admits a helix join.

Graphs that are 2-connected but not 3-connected

Let G be a helix join of graphs, G_1 and G_2.

Lemma (B and Brettell, 2023+)

If G_1 and G_2 are 3-unrestricted then G is 3-unrestricted.

Lemma (B and Brettell, 2023+)

If G_1 and G_2 are 3-restricted on certain vertices involved in the helix join then G is 3-restricted.

Where to Next?

We wish to describe precisely the cases where a graph has maximal local edge connectivity 3 and is not 3-choosable.

Conjecture

Let G be a helix join of graphs, G_1 and G_2. If G_1 is 3-unrestricted and G_2 is critically 3-restricted then G is 3-unrestricted.

Then work out which vertices are restricted in our restricted class of graphs. Then solve the problem for $k = 3$.

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Thanks for your attention.