On linear-algebraic notions of expansion

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Centre for Quantum Software and Information
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For more details, please refer to ArXiv:2212.13154.

Outline

- Introduction about graph-theoretic expansion and a classical result.
- Introduction about linear-algebraic expansion and some previous results.
- An overview of our main results.
  1. Dimension expansion $\not\Rightarrow$ Quantum expansion.
  2. Quantum expansion $\Rightarrow$ Dimension expansion.
  3. Linear-algebraic expansion properly generalizes graph-theoretic expansion.

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What are expanders?

- Expanders are graphs that are simultaneously sparse and highly connected.

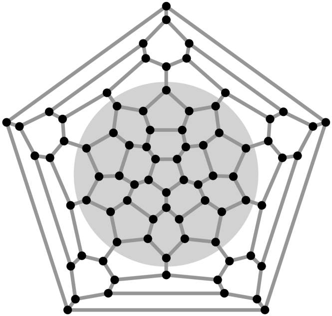

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Figure source: `https://www.ams.org/notices/200407/what-is.pdf`

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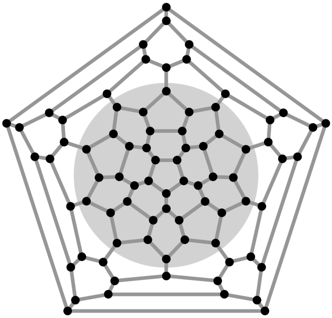

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Definition of graph-theoretic expansion

- Let $G = ([n], E)$ be a d-regular graph.
- The **spectral expansion** of G:

  $$\lambda(G) := \text{the second-largest absolute value over all eigenvalues of } A,$$

  where A is the normalized adjacency matrix of G.
- The largest absolute value over all eigenvalues of A is 1!
- The larger the spectral gap $1 - \lambda(G)$ is, the better the expansion is.

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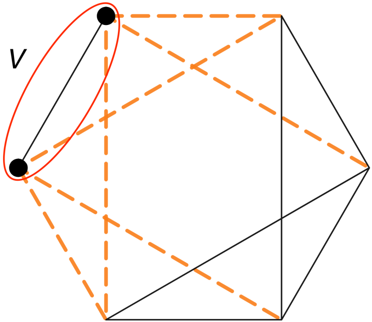
Examples of graph expansion



Edge expansion for vertex subset $V = \frac{6}{2} = 3$

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- Let $G = ([n], E)$ be a d-regular graph.
- The **edge expansion** of G:

$$h(G) := \min_{\substack{V \subseteq [n] \\ 1 \leq |V| \leq \frac{n}{2}}} \frac{|\partial(V)|}{|V|},$$

where $\partial(V) := \{\{i,j\} \in E : i \in V, j \in [n] \setminus V\}$.

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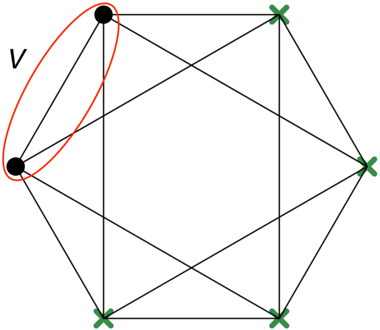

Vertex expansion for vertex subset $V = \frac{4}{2} = 2$

Definition of graph-theoretic expansion

- Let $G = ([n], E)$ be a d-regular graph.
- The **vertex expansion** of G:

$$\mu(G) := \min_{\substack{V \subseteq [n] \\ 1 \leq |V| \leq \frac{n}{2}}} \frac{|\partial_{\text{out}}(V)|}{|V|},$$

where $\partial_{\text{out}}(V) := \{j \in [n] \setminus V : \exists i \in V, \text{ s.t. } \{i, j\} \in E\}$.

A classical result of their relationship

Recall that

- λ: spectral expansion
- h: edge expansion
- μ: vertex expansion

For any d-regular graph G, the three notions of expansion are all equivalent, in the sense that

- $\frac{\mu(G)}{d} \leq h(G) \leq \mu(G)$ (By definition);
- $\frac{1-\lambda(G)}{2} \leq h(G) \leq \sqrt{2(1-\lambda(G))}$ (discrete Cheeger's inequality)

  [Dodziuk'84, Alon-Milman'85, Alon'86]

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  [Dodziuk'84, Alon-Milman'85, Alon'86]

Definition of some linear-algebraic expansion

- Given a matrix tuple $\mathbf{B} = (B_1, \ldots, B_d) \in \mathrm{M}(n, \mathbb{C})^d$.
- $\mathbf{B}$ is a *doubly stochastic matrix tuple* if $\sum_{i=1}^d B_i B_i^* = \sum_{i=1}^d B_i^* B_i = dI_n$.
- The *associated quantum operator* is the linear map $\Phi_{\mathbf{B}} : \mathrm{M}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$ defined by

$$\Phi_{\mathbf{B}}(X) := \frac{1}{d} \sum_{i=1}^{d} B_i X B_i^*.$$

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Definition of some linear-algebraic expansion

- The **quantum expansion** of $\Phi_{\mathbf{B}}$ [Ben–Aroya-Ta–shma'07, Hastings'07]:

  $\lambda(\mathbf{B}) :=$ the second-largest absolute value over all eigenvalues of $\Phi_{\mathbf{B}}$.

- The **quantum edge expansion** of $\Phi_{\mathbf{B}}$ [Hastings'07]:

$$h_Q(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V) \rangle}{\dim(V)},$$

  where P_V is the orthogonal projection to the subspace $V \leq \mathbb{C}^n$.

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Intuition of the definition

edge expansion **quantum edge expansion**

$$\min_{\substack{V\subseteq[n]\\ 1\leq|V|\leq\frac{n}{2}}} \frac{|\partial(V)|}{|V|} \quad \Longrightarrow \quad \min_{\substack{V\leq\mathbb{C}^n\\ 1\leq\dim(V)\leq\frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle}{\dim(V)}$$

- $\Phi_{\mathbf{B}}$ is an analogue of the normalized adjacency matrix A of a graph G.
- Consider that all the B_i's are permutation matrices.
- Consider that V is a coordinate subspace. (So P_V is diagonal of 0 and 1!)
- Then $I_n - P_V$ can be treated as an indicator vector x, and $\Phi_{\mathbf{B}}(P_V) = Ax$.
- So $\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle$ counts the edges between a set and its complement.

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$$\min_{\substack{V\subseteq[n]\\ 1\leq|V|\leq\frac{n}{2}}} \frac{|\partial(V)|}{|V|} \quad ====\Rightarrow \quad \min_{\substack{V\leq\mathbb{C}^n\\ 1\leq\dim(V)\leq\frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle}{\dim(V)}$$

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  - A d-regular graph can be decomposed as a union of d permutations.
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Definition of some linear-algebraic expansion

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$$\mu(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\dim(V + \mathbf{B}(V)) - \dim(V)}{\dim(V)},$$

where $\mathbf{B}(V) := \left\langle \cup_{i \in [d]} \{B_i v : v \in V\} \right\rangle$.

- Given the **vertex expansion** $\mu(G)$ and treat G as a tuple of permutations $(P_1, \ldots, P_d)$ acting on $[n]$:

$$\forall V \subseteq [n], \left| V \cup \bigcup_{i=1}^{d} P_i(V) \right| \geq (1 + \mu(G))|V|$$

- Change the permutation action and underlying object to be more general.

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where $V^\perp$ means the orthogonal complement of V, and the columns of T_V form an orthonormal basis of V.

- Let $\dim(V) = r$ and $U = \begin{bmatrix} T_V & T_{V^\perp} \end{bmatrix}$ be an $n \times n$ unitary matrix.

$$U^* B_i U = \begin{bmatrix} * & * \\ T_{V^\perp}^* B_i T_V & * \end{bmatrix},$$

where $T_{V^\perp}^* B_i T_V \in \mathrm{M}((n-r) \times r, \mathbb{C})$.

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vertex expansion $\Longrightarrow$ **dimension expansion**

edge expansion $\Longrightarrow$ **dimension edge expansion**

- If we restrict
  - the matrix tuple consisting of permutation matrices only and;
  - the minimum to coordinate subspaces only,

  one can precisely recover the definition of corresponding graph expansion.

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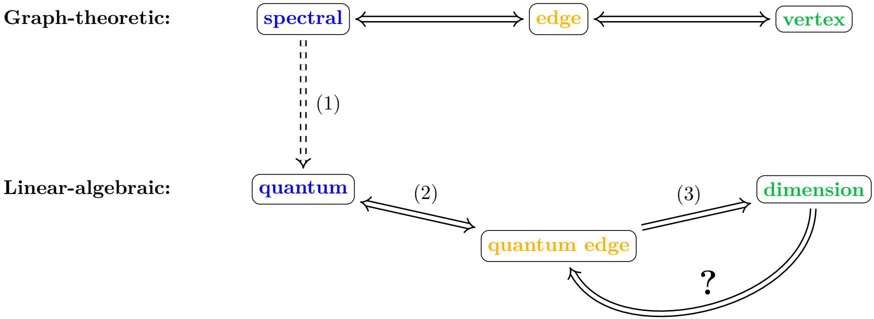
Overview of previous results



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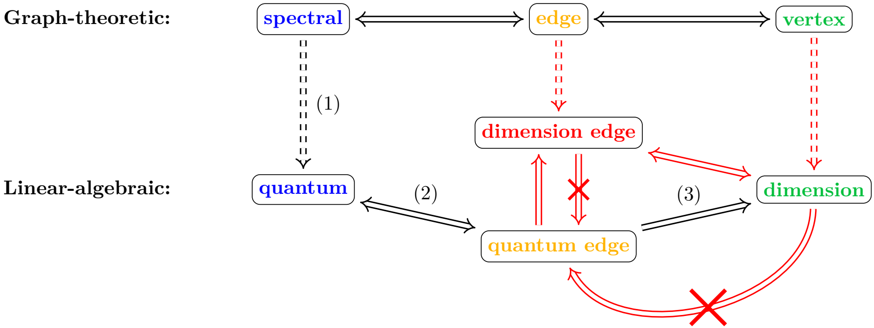
Overview of our main results



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