The Distinguishing number of bipartite 2-arc-transitive graphs

Joint work with Alice Devillers, Luke Morgan and Friedrich Rober

Lei Chen

The University of Western Australia

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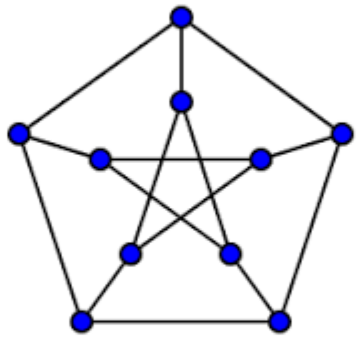
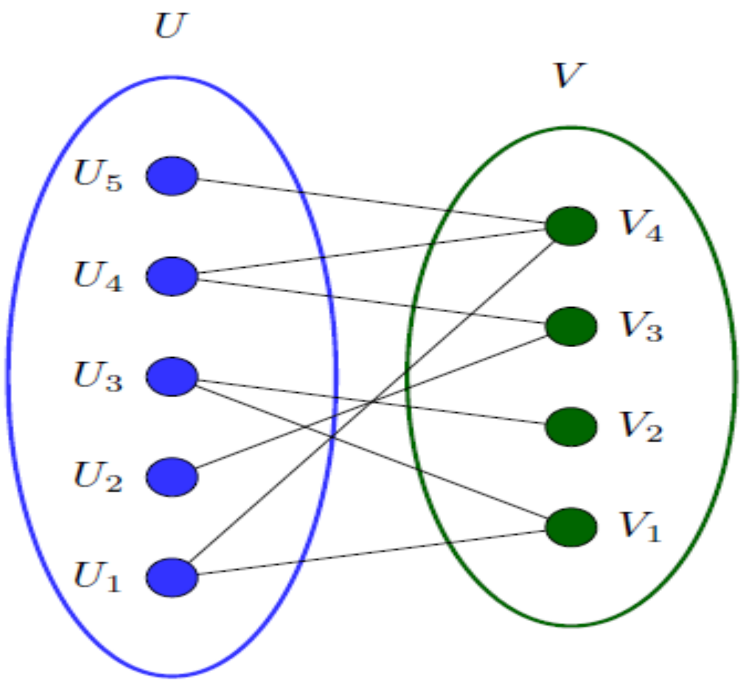


Bipartite Graph

- *A bipartite graph is a graph whose vertex set can be divides into 2 disjoint sets such that no two graph vertices within the same set are adjacent.*

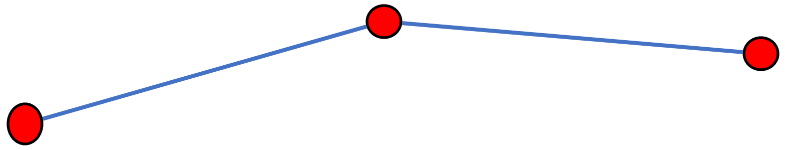
- *A bipartite graph does not contain odd cycles*

- *The Petersen graph is not a bipartite graph*



Properties of Graphs

- *A graph Γ is said to be* *connected* *if for any two vertices u and v in Γ there exists a path u→… →v.*
- *A graph Γ is said to be* *vertex-transitive* *if Aut(Γ) acts transitively on the vertex set of Γ.*
- *A graph Γ is said to be* *2-arc-transitive* *if Aut(Γ) acts transitively on the set of 2-arcs {u →v →w} of Γ.*



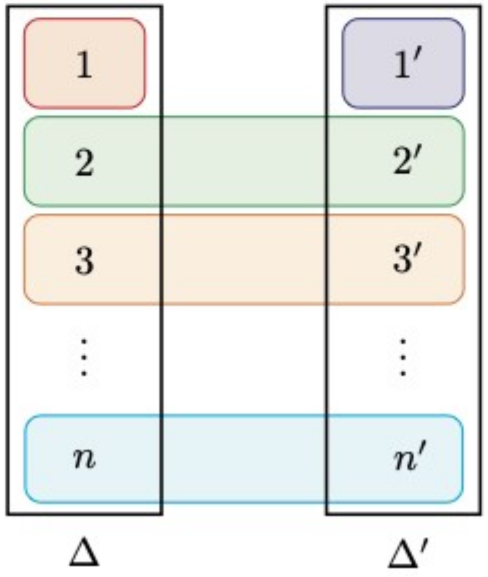
Distinguishing number (of a graph)

- *The distinguishing number $D(\Gamma)$ of a graph is the least integer such that has a vertex labeling with labels that is preserved only by a trivial automorphism.*

- *The distinguishing number $D_A(G)$ of a group G acting on a set A is the least integer k such that there exists a partition Π of A into k parts such that only the identity of G stabilises each part.*

Distinguishing number for $K_{n,n}$

- *Let Δ and Δ’ be two parts of Γ, then for every u in Δ and every v in Δ’ , there exists an arc between u and v.*
- *G=Aut(Γ) is isomorphic to Sym(n) wr Sym(2)*
- *$G=G^+$:<x>, where x is an involution swapping Δ and Δ’ and $G^+=H \times K$, where H and K are isomorphic to Sym(n)*
- *The induced action of G^+ on Δ is Sym(n)*
- *D(G)=n+1*



Motivation

- *Devillers, Morgan, Harper: The distinguishing number of quasiprimitive(all of the non-trivial normal subgroups are transitive) and semiprimitive groups*
- *Investigate the distinguishing number of non-bipartite graphs*
- *Most of them have distinguishing number 2*
- *If $\Gamma \neq K_n$ and G quasiprimitively act on Γ such that G is not $Sym(n)$ or $Alt(n)$, then $D(G)<5$.*



Result of Seress

- *If X is a primitive group of degree n and Alt(n) is not contained in X, then either D(X)=2, or (X, n) is listed below*

TABLE 1. The affine groups $X \leqslant \mathrm{Sym}(n)$ in the set $\mathcal{P}$

Groups with $D(X) = 3$					
ID	$[5,2]$	$[5,3]$	$[7,4]$	$[8,2]$	$[9,4]$
X	D_{10}	F_{20}	F_{21}	$2^3.F_{21}$	$3^2.D_8$
	$[9,5]$	$[9,6]$	$[9,7]$	$[16,16]$	$[16,17]$
	$3^2.8.2$	$3^2.\mathrm{SL}(2,3)$	$3^2.\mathrm{GL}(2,3)$	$2^4.\Gamma\mathrm{L}(2,4)$	$2^4.\mathrm{Alt}(6)$
	$[16,18]$	$[16,19]$	$[16,20]$	$[32,3]$	
	$2^4.\mathrm{Sym}(6)$	$2^4.\mathrm{Alt}(7)$	$2^4.\mathrm{Alt}(8)$	$2^5.\mathrm{GL}(5,2)$	
Groups with $D(X) = 4$					
ID	$[8,3]$				
X	$2^3.\mathrm{GL}(3,2)$				

TABLE 2. The almost simple groups $X \leqslant \mathrm{Sym}(n)$ in the set $\mathcal{P}$

Groups with $D(X) = 3$					
ID	$[6,1]$	$[8,4]$	$[8,5]$	$[9,8]$	$[9,9]$
X	$\mathrm{PSL}(2,5)$	$\mathrm{PSL}(2,7)$	$\mathrm{PGL}(2,7)$	$\mathrm{PSL}(2,8)$	$\mathrm{P\Gamma L}(2,8)$
	$[10,2]$	$[10,3]$	$[10,4]$	$[10,5]$	$[10,6]$
	$\mathrm{Sym}(5)$	$\mathrm{Alt}(6)$	$\mathrm{Sym}(6)$	$\mathrm{Alt}(6).2_2$	$\mathrm{Alt}(6).2_3$
	$[10,7]$	$[11,5]$	$[12,2]$	$[12,3]$	$[13,7]$
	$\mathrm{Alt}(6).2^2$	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$	M_{11}	$\mathrm{PSL}(3,3)$
	$[14,2]$	$[15,4]$	$[17,7]$	$[17,8]$	$[21,7]$
	$\mathrm{PGL}(2,13)$	$\mathrm{Alt}(8)$	$\mathrm{PSL}(2,16).2$	$\mathrm{P\Gamma L}(2,16)$	$\mathrm{P\Gamma L}(3,4)$
	$[22,1]$	$[22,2]$	$[23,5]$	$[24,3]$	
	M_{22}	$\mathrm{M}_{22}.2$	M_{23}	M_{24}	
Groups with $D(X) = 4.$					
ID	$[6,2]$	$[7,5]$	$[11,6]$	$[12,4]$	
X	$\mathrm{PGL}(2,5)$	$\mathrm{PSL}(3,2)$	M_{11}	M_{12}	

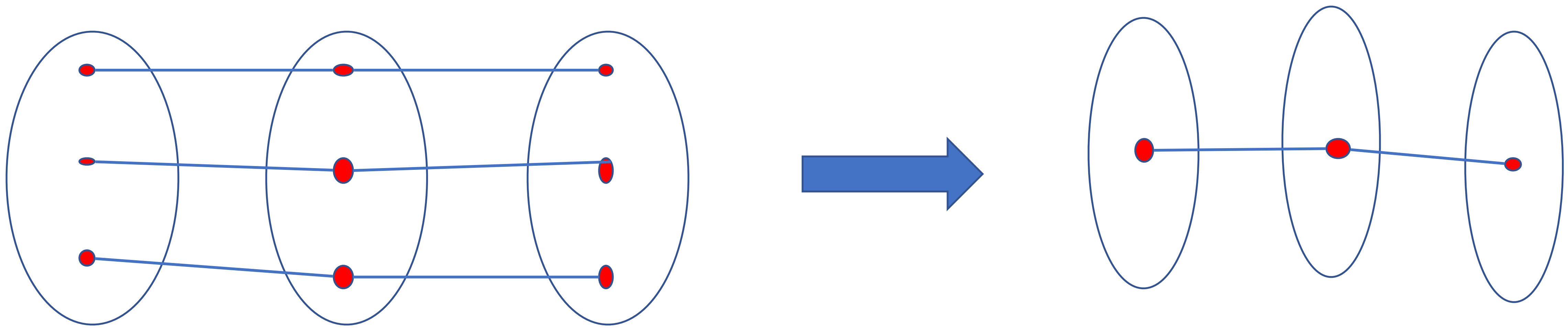
Quotient graph

Input: connected graph Γ=(V,E) and

- *G is contained in Aut(Γ), G transitive on E*
- *N normal subgroup of G*

Output: normal quotient Γ/N=(V/N,E/N)

- *V/N set of N-orbits in V*
- *E/N N-orbit pairs connected by at least 1 edge*



Results about the quotient graph of a bipartite graph

- *Praeger: Γ a G-vertex-transitive (G,2)-arc-transitive bipartite graph, if N is an intransitive subgroup of G with more than 2 orbits on V Γ, then N is semiregular (i.e. the stabiliser of any point x in N is trivial). Moreover, the quotient graph is also (G/N,2)-arc-transitive bipartite graph.*

- *Devillers-Morgan-Harper 2018: Let G act transitively on a set Γ and N be a semiregular (i.e. the stabiliser of any point x in N is trivial) and intransitive subgroup of G, then D(G)≤ D(G/N).*

Our strategy

- *Need to find a normal subgroup N of G contained in G^+ (an index 2 subgroup of G which has exactly two orbits Δ and Δ') such that N is intransitive on both Δ and Δ' . Then N has at least 2 orbits on each side of Γ and therefore there are more than 3 orbits.*

- *Pick N carefully so that G^+ /N acts quasiprimitively on Δ/N and Δ'/N.*

- *Note that D(G)≤ D(G/N).*

Theorem

Let Γ be a G-vertex-transitive (G,2)-arc-transitive bipartite graph and N be an intransitive normal subgroup of G contained in G^+ that is maximal subject to being intransitive on both Δ and Δ'. Then N is semiregular on $V\Gamma$. If $D(G^\Gamma)>2$, then

- *N is normal in G*

○ (1) *Γ/N is $K_{n,n}$, G^+/N is primitive on both Δ/N and Δ'/N*

○ *or* (2) *G^+/N is faithful and quasiprimitive on both Δ/N and Δ'/N.*

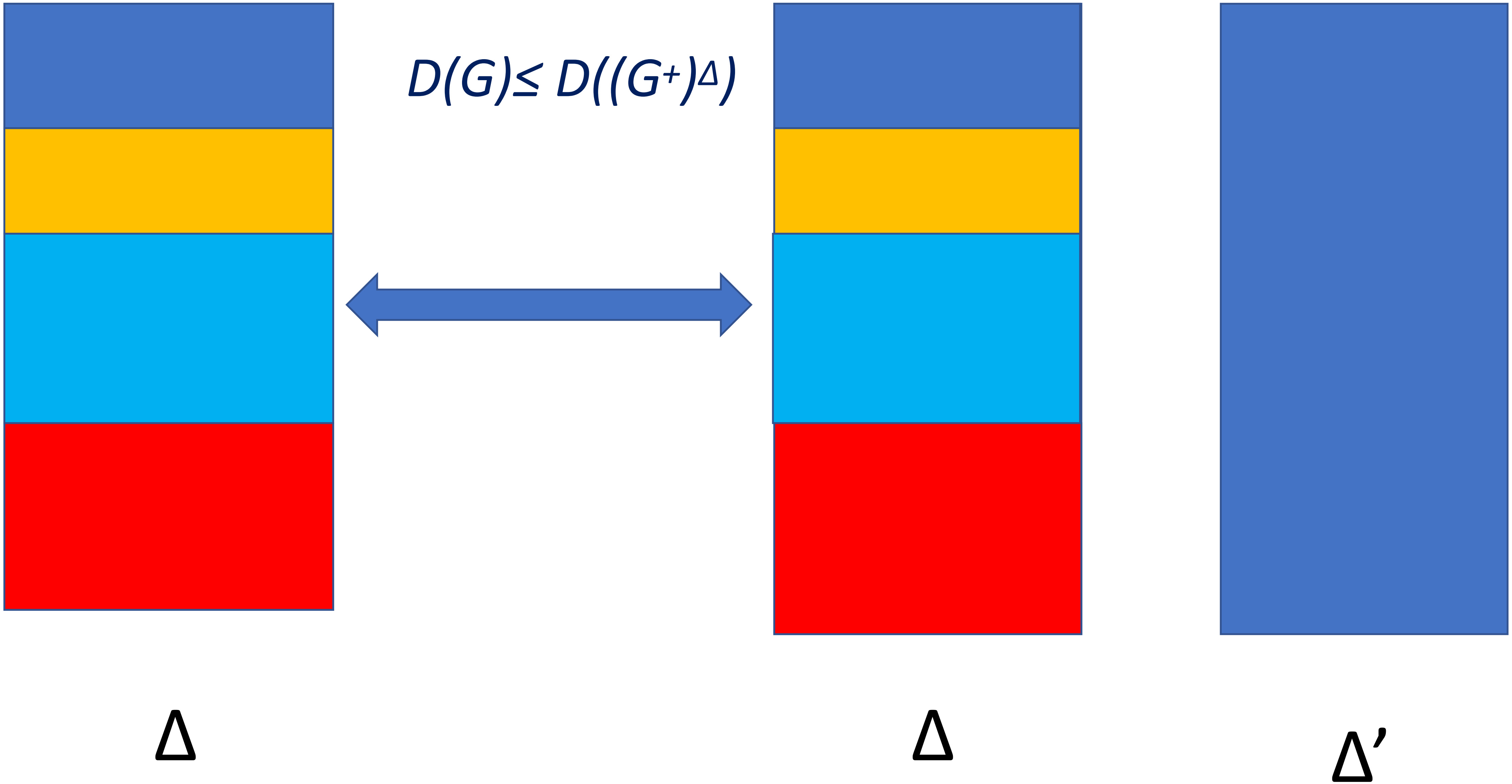
Case (2) of Theorem

- *Faithfulness of G^+/N implies that*

$$G^+/N = \{(x,y) \mid x \text{ in } H,\ y \text{ in } K\},$$

where H acts on Δ/N and K acts on Δ'/N, where $x=y^f$, where f is some automorphism of H.

- *This implies that once the symmetry on Δ/N is broken, the symmetry on Δ'/N is also broken.*


$D(G) \leq D((G^+)^\Delta)$
Δ
Δ
Δ'

Numerical Bound

Let Γ be a G-vertex-transitive (G,2)-arc-transitive bipartite graph

- *Take the normal quotient, then we have $D(G) \leq D(G/N)$*
- *If G^+/N is not containing Alt(n), then G^+/N acts faithfully on both $\Pi := \Delta/N$ and $\Pi' := \Delta'/N$ and so $D(G/N) \leq D((G^+/N)^{\Pi})$*
- *Since G^+/N also acts quasiprimitively on Π, we know that $D((G^+/N)^{\Pi})$ is bounded by 4*
- *Hence $D(G) \leq 4$*

Main Theorem

Theorem 1.1. *Let Γ be a bipartite connected G-vertex-transitive, $(G,2)$-arc transitive graph with $G \leqslant \mathrm{Aut}(\Gamma)$. If $D(G) \geqslant 3$ then one of the following holds:*

1. $\Gamma = K_{6,6}$, $G^+ = \mathrm{Diag}_\varphi(H \times H)$ *where* $H = \mathrm{Alt}(6)$ *or* $\mathrm{Sym}(6)$ *and* φ *is an outer automorphism of* $\mathrm{Sym}(6)$, *and* $D(G) = 3$;
2. $\Gamma = K_{n,n} - nK_2$, $G = \mathrm{Alt}(n) \times C_2$ *and* $D(G) = \lceil \sqrt{n-1} \rceil$ *with* $n \geqslant 6$;
3. $\Gamma = K_{n,n} - nK_2$, $G = \mathrm{Sym}(n) \times C_2$ *and* $D(G) = \lfloor \sqrt{n} \rfloor + 1$ *with* $n \geqslant 4$;
4. $\Gamma = K_{n,n}$, *and*

$$D(G) = \begin{cases} n-1 & \text{if } G^+ = \mathrm{Alt}(n) \times \mathrm{Alt}(n) \\ n & \text{if } G^+ = \langle \mathrm{Alt}(n) \times \mathrm{Alt}(n), ((1,2),(1,2)) \rangle \\ n+1 & \text{if } G^+ = \mathrm{Sym}(n) \times \mathrm{Sym}(n) \end{cases}$$

5. $\Gamma = K_{8,8} - 8K_2$, $G \cong \mathrm{AGL}(3,2) \times C_2$ *and* $D(G) = 3$;
6. $\Gamma = K_{8,8}$, $G = \mathrm{AGL}(3,2) \wr C_2$ *and* $D(G) = 5$;

REFERENCE

- *C.E. Praeger, Finite transitive permutation groups and bipartite vertex-transitive graphs, Illinois Journal of Mathematics, 47 (2003), 461-475*

- *A. Seress, Primitive groups with no regular orbits on the set of subsets, Bulletin of the London Mathematical Society, 29 (1997), 697-704*

- *A.Devillers, L. Morgan and S. Harper, The distinguishing number of quasiprimitive and semiprimitive groups, Archiv der Mathematik 113.2 (2019), 127-139.*

- *M. Giudici, C. Li and C.E. Praeger, Analysing finite locally s-arc-transitive graphs, Transactions of the American Mathematical Society 356 (2004).*