Diameter of some families of quotient-complete, arc-transitive graphs

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45th ACC

Preliminaries

Given **graph** $\Gamma = (V(\Gamma), E(\Gamma))$

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G-**arc-transitive:** G is transitive on ordered pairs of adjacent vertices (arcs)

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- $S =$ neighbors of 0_V
- Γ is connected $\iff \langle S \rangle = V$
- $\text{diam}(\Gamma) \leq n \iff V \subseteq S \cup (S + S) \cup \ldots \cup \underbrace{(S + S + \ldots + S)}_{n \text{ copies}}$

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Example: $\Gamma = \text{Cay}(V, S)$

- $V = \mathbb{F}_3 \oplus \mathbb{F}_3$
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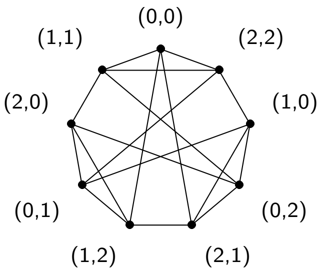

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Some properties:

- $\mathrm{diam}(\Gamma_N) \leq \mathrm{diam}(\Gamma)$
- Γ connected $\Rightarrow \Gamma_N$ connected
- Γ G-vertex-transitive $\Rightarrow \Gamma_N$ G/N-vertex-transitive
- Γ G-arc-transitive $\Rightarrow \Gamma_N$ G/N-arc-transitive

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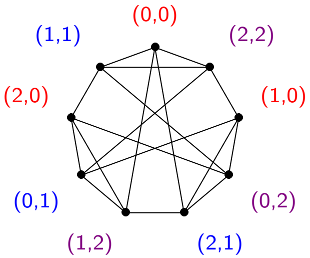

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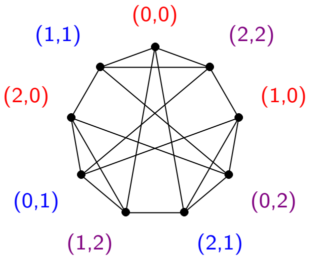

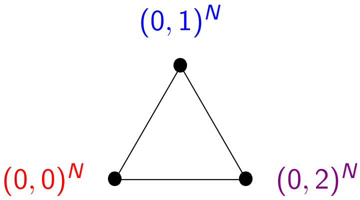

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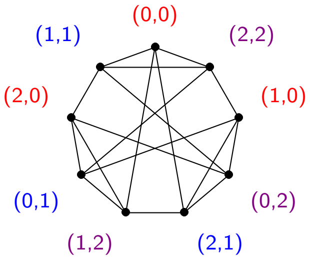

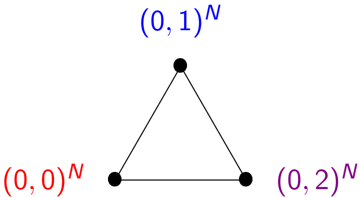

$\Gamma_N \cong$ complete graph of order 3

Background

Normal quotient reduction:

arc-transitive, diameter two graphs

$\exists\ N \trianglelefteq G$

Γ_N vertex-quasiprimitive, $\mathrm{diam}(\Gamma_N) = 2$	Γ_N quotient-complete, $\mathrm{diam}(\Gamma_N) = 2$
proper normal quotients are trivial	proper normal quotients are complete or empty

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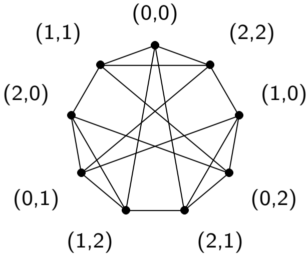

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Four $N \triangleleft G$: translations by $\mathbb{F}_3 \oplus \{0_V\}, \{0_V\} \oplus \mathbb{F}_3, \langle (1,1) \rangle$, and $\langle (1,2) \rangle$

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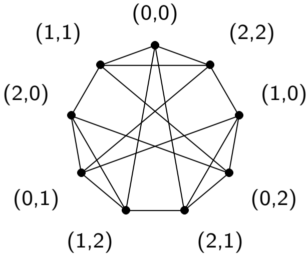

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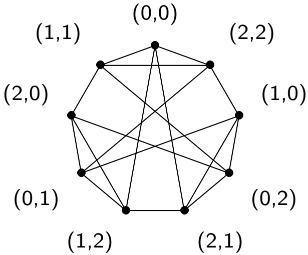
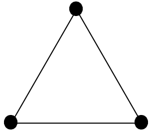


$\longrightarrow$

Γ_N:



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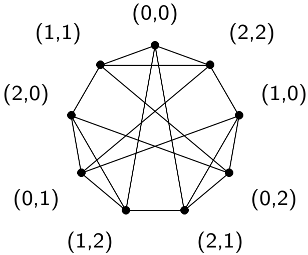

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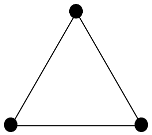

Γ is quotient-complete with $k = 4$

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- $\exists$ at least one proper normal quotient Γ_N that is nontrivial and complete, and
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General problem:

Classify arc-transitive, quotient-complete, diameter-two graphs

Classification status

$k \geq 3$: $H \leq GL(V)$

Case $H \not\leq \Gamma L_1(q)$: (Amarra, Giudici, Praeger, 2012)

Case $H \leq \Gamma L_1(q)$: (De Vera, MSc Thesis 2021)

Problem 1

✓diam 2, $H \not\leq \Gamma L_1(q)$

? diam 2, $H \leq \Gamma L_1(q)$

Quotient-complete arc-transitive Γ

$k = 1$ or 2

some examples known

Problem 2

? diam 2

$k :=$ number of proper normal quotients

Quotient-complete, arc-transitive, $k \geq 3$

Theorem [Amarra, Giudici, Praeger, 2012].

Γ *is* G*-arc-transitive,* G*-quotient-complete, with at least 3 nontrivial complete* G*-normal quotients* $\Longrightarrow$

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Unknown: Diameter two Γ when $H \leq \Gamma L_1(q)$.

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$\Gamma L_1(q) = \langle \tau, \hat{\omega} \rangle$, $q = p^d$, τ is the Frobenius automorphism,
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F1. $c > 0$ *and* $c \mid p^d - 1$,

F2. $s > 0$ *and* $s \mid d$, *and*

F3. $0 \leq e < c$ *and* $c \mid e(p^d - 1)/(p^s - 1)$.

F4. $e > 0$ *and* $c \mid e(p^{cs} - 1)/(p^s - 1)$, *and*

F5. *if* $1 < c' < c$ *then* $c \nmid e(p^{c's} - 1)/(p^s - 1)$.

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- f1. $c > 1$,
- f2. $cs \mid d$,
- f3. $0 < e < c$ *and* $\gcd(c, e) = 1$,
- f4. $p^s \equiv 1 \pmod{d'}$ *for every prime divisor d' of c, and*
- f5. $p \equiv 1 \pmod{4}$ *whenever $4 \mid c$ and s is odd.*

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- diameter 3, if $\lambda \in \{\omega, \omega^3, \omega^4, \omega^{12}, \omega^{13}, \omega^{31}, \omega^{41}, \omega^{43}, \omega^{44}, \omega^{52}, \omega^{53}, \omega^{71}\}$

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- diameter 2, if $\lambda \in \{\omega^2, \omega^5, \omega^6, \dots\}$ (24 graphs)

Classification status

$k \geq 3$: $H \leq GL(V)$

Case $H \not\leq \Gamma L_1(q)$: (Amarra, Giudici, Praeger, 2012)

Case $H \leq \Gamma L_1(q)$: (De Vera, MSc Thesis 2021)

Problem 1

✓diam 2, $H \not\leq \Gamma L_1(q)$

? diam 2, $H \leq \Gamma L_1(q)$

Quotient-complete arc-transitive Γ

$k = 1$ or 2

some examples known

Problem 2

? diam 2

$k :=$ number of proper normal quotients

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Some G-arc-transitive, G-quotient-complete graphs with $k = 1$ or 2:

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3. (Amarra, 2018) Some latin square graphs from Cayley table of elementary abelian groups has $k = 1$ or 2

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N_1: translations by elements of $\mathbb{F}_q \oplus \{0\}$

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Theorem (Main Result 2).

r	q	$\mathrm{diam}(\Gamma)$
1	≥ 3	*2*
2	≥ 5	*2*
$3 \leq r < q - 1$	≥ 5	*3 or 4*
$3 \leq r = q - 1$	≥ 5	*2 or 3*

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r	q	$\text{diam}(\Gamma)$	
1	≥ 3	2	
2	≥ 5	2	
$3 \leq r < q-1$	≥ 5	3 or 4	*which have*
$3 \leq r = q-1$	≥ 5	2 or 3	$\text{diam}(\Gamma) = 2$?

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Corollary

If $r \leq d$ then $\mathrm{diam}(\Gamma) = 2$.

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