The Hamilton decomposition problem

45th Australasian Combinatorics Conference

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Australia

A survey talk, including some joint work with:

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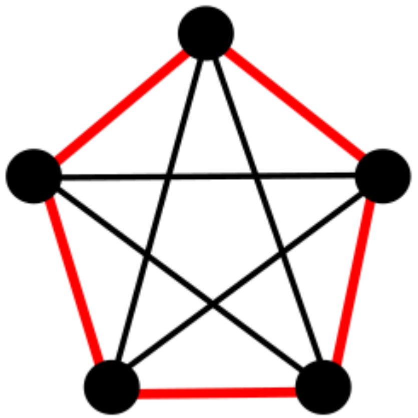
Outline for this talk

- Definitions and historical context
- A brief overview of the Hamiltonian problem
- Hamilton decompositions of…
  - Complete multipartite graphs
  - Vertex-transitive graphs
  - Cayley graphs and infinite Cayley graphs
  - Graph products
  - Line graphs

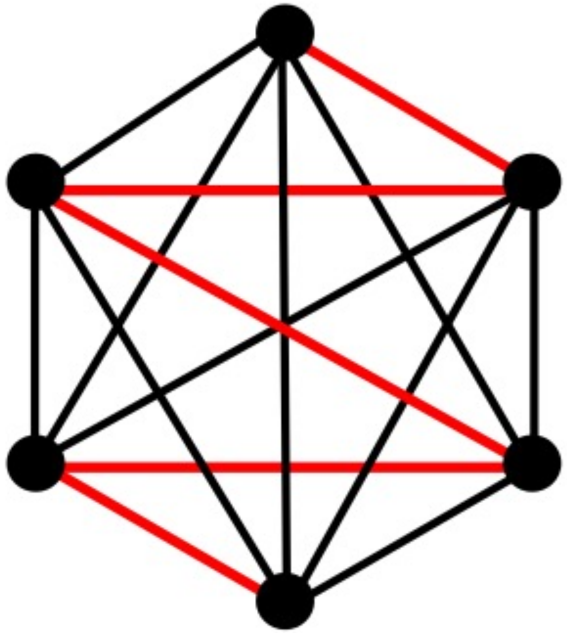
Hamilton cycles and paths

A Hamilton cycle is a cycle that contains every vertex of a graph.
If a graph is Hamiltonian if it has a Hamilton cycle.



Determining whether a general graph is Hamiltonian
is an **NP-complete problem**.

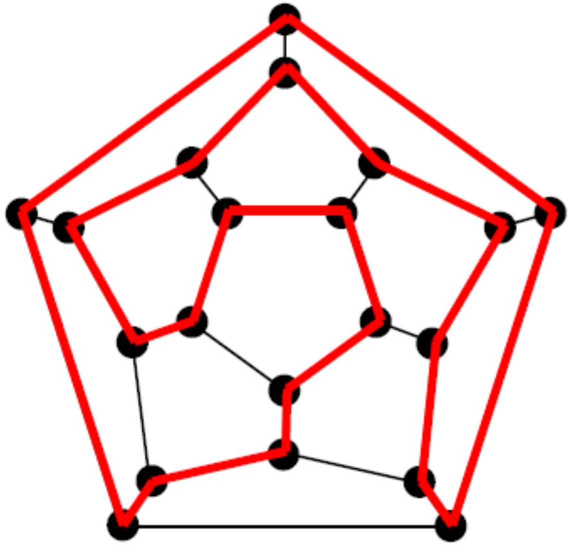
A Hamilton path is a path that contains every vertex of a graph.



Historical context

The name *Hamilton* acknowledges mathematician Sir William Rowan Hamilton who introduced the Icosian Game in 1857.

https://www.puzzlemuseum.com/



Kirkman (1855) *Given the graph of a polyhedron, can one always find a circuit that passes through each vertex exactly once?*

50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	64	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	1	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

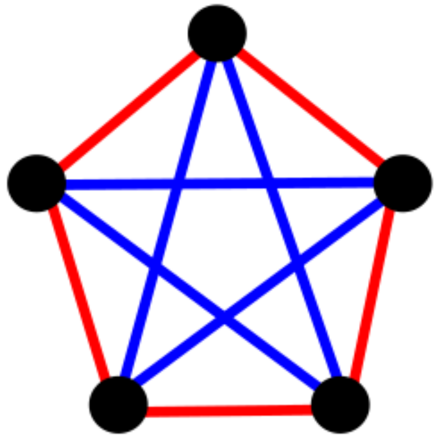
Knight’s tour problem

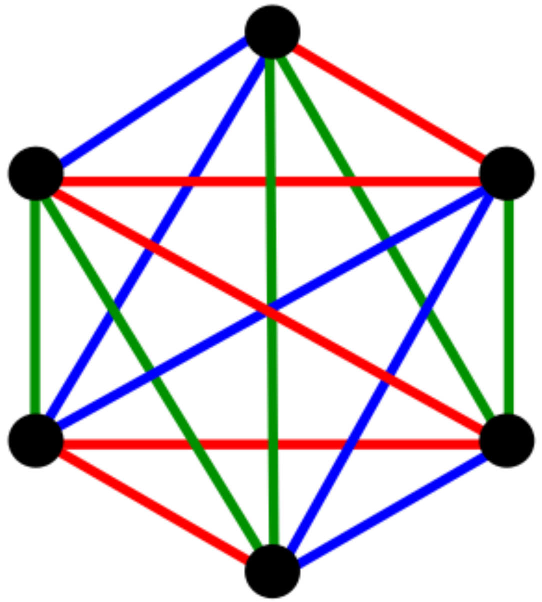
Decompositions

A decomposition of a graph G is a set $\{H_1, H_2, \ldots, H_t\}$ of edge-disjoint subgraphs of G such that $E(H_1) \cup \mathrm{E}(H_2) \cup \ldots \cup \mathrm{E}(H_t) = \mathrm{E}(G)$.

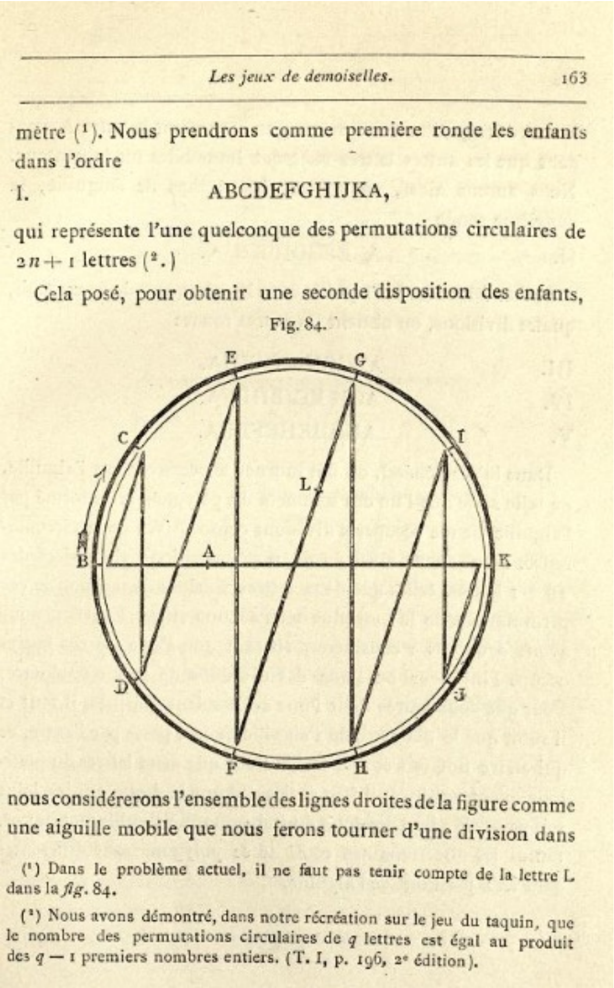
A Hamilton decomposition is a decomposition into Hamilton cycles.

A Hamilton path decomposition is a decomposition into Hamilton paths.





Decompositions of complete graphs



mètre (¹). Nous prendrons comme première ronde les enfants dans l'ordre

I. ABCDEFGHIJKA,

qui représente l'une quelconque des permutations circulaires de $2n+1$ lettres (²).

Cela posé, pour obtenir une seconde disposition des enfants,

Fig. 84.

nous considérerons l'ensemble des lignes droites de la figure comme une aiguille mobile que nous ferons tourner d'une division dans

(¹) Dans le problème actuel, il ne faut pas tenir compte de la lettre L dans la *fig.* 84.

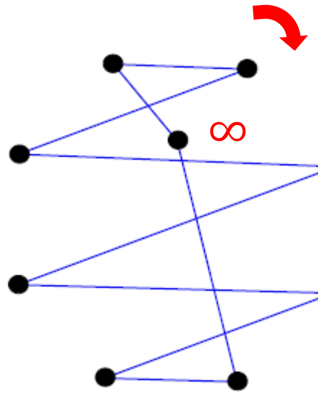
(²) Nous avons démontré, dans notre récréation sur le jeu du taquin, que le nombre des permutations circulaires de q lettres est égal au produit des $q-1$ premiers nombres entiers. (T. I, p. 196, 2e édition).

Image from
Lucas, Récréations mathématiques (1892) vol 2

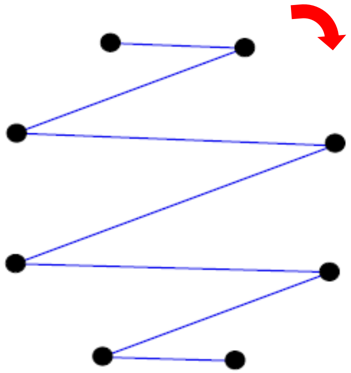
Walecki (1892)

K_n decomposes into Hamilton cycles $\Leftrightarrow$ n is odd.

K_n decomposes into Hamilton paths $\Leftrightarrow$ n is even.



K_9 into 4 Hamilton cycles



K_8 into 4 Hamilton paths

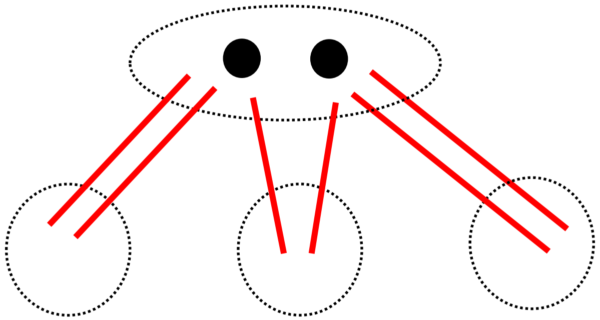
A brief overview of the Hamiltonian problem

Classic results

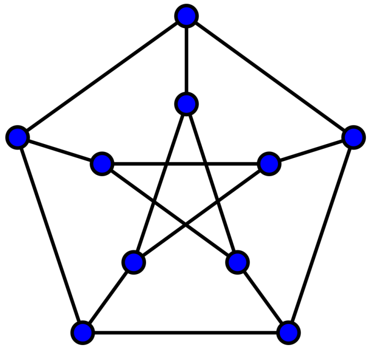
Dirac's Theorem (1952) If G is a graph of order $n \geq 3$ and minimum degree at least $\frac{n}{2}$ then G is Hamiltonian.

Ore's Theorem (1960) If G is a graph of order $n \geq 3$ and for every pair of non-adjacent vertices u and v, we have $\deg(u) + \deg(v) \geq n$, then G is Hamiltonian.

Consider a graph with a non-empty subset of vertices S whose removal results in more than $|S|$ components.



A graph is 1-tough if it does <u>not</u> have such a set.



1-tough but non-Hamiltonian

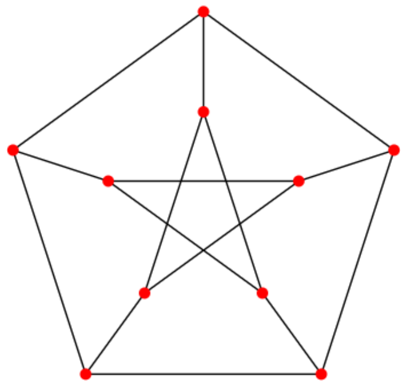
Theorem (Chvátal 1973) Every Hamiltonian graph is 1-tough.

Vertex-transitive graphs

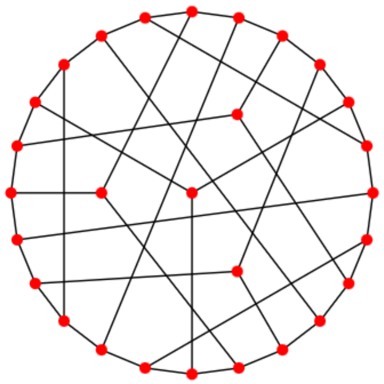
Conjecture (Lovász 1969): Every finite connected vertex-transitive graph has a Hamilton path.

Conjecture: Every finite connected vertex-transitive graph is Hamiltonian, with 4 nontrivial exceptions.

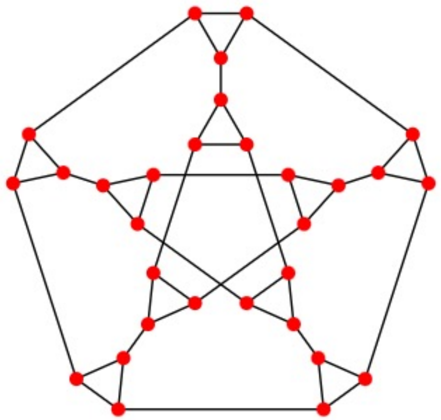
Thomassen conjectured only finitely many exceptions; Babai conjectured infinitely many.



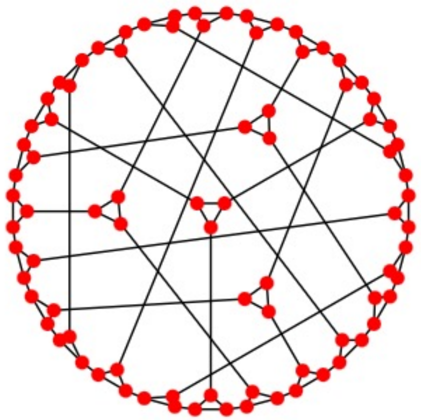
Petersen graph



Coxeter graph



triangle-replaced Petersen graph



triangle-replaced Coxeter graph

https://mathworld.wolfram.com/NonhamiltonianVertex-TransitiveGraph.html

Vertex-transitive graphs

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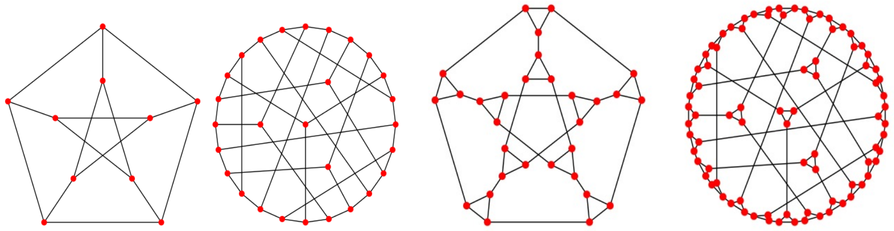
- ✓ Order $p \geq 3$ — Turner [1976]
- ✓ Order kp for $k \leq 4$ — Alspach [1979] Marušič [1988] Kutnar & Marušič [2008]
- ✓ Order p^j for $j \leq 4$ — Marušič [1985] Chen [1996] Zhang [2015]
- ✓ Order pq — Du, Kutnar and Marušič [2021]

Kneser graphs $K(n,k)$ are Hamiltonian (except Petersen) — Merino et al. [2023+, arXiv]

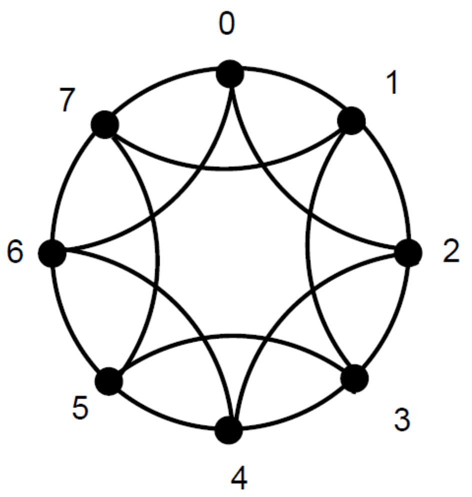
Except for:



Cayley graphs

Let $(\mathrm{X}, +)$ be a group with identity e and $\mathrm{S} \subseteq X - \{e\}$ be inverse-closed.

The Cayley graph on the group X with connection set S, denoted $\mathrm{Cay}(\mathrm{X}, \mathrm{S})$, is the graph with vertex set X and edge set $\{\{x, x+s\} : x \in \mathrm{X}, s \in \mathrm{S}\}$.



$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$

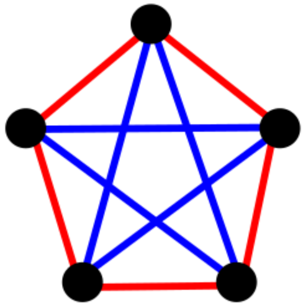
Folklore Conjecture
Every finite connected Cayley graph of order at least 3 is Hamiltonian.

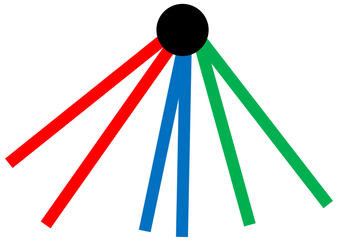
- ✓ X is an abelian group. — known by Lovasz [1979]
- ✓ X has prime power order greater than 2. — Witte [1986]
- ✓ X is the dihedral group D_n with n even. — Alspach, Chen, Dean [2010]
- ✓ Almost all Cayley graphs are Hamiltonian. — Jixiang, Qiongxiang [1996]

Hamilton decompositions

Hamilton decompositions

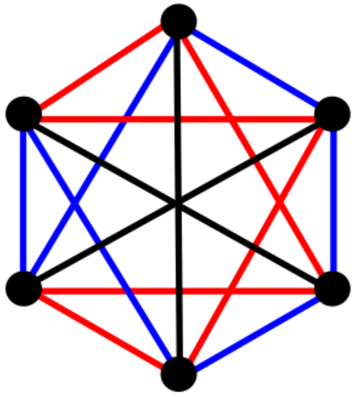
A **Hamilton decomposition** is a decomposition into Hamilton cycles.

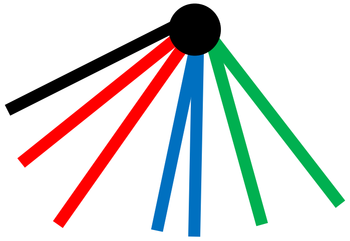




regular of
even degree

If G is regular of **odd degree**, then a **Hamilton decomposition** of G is a decomposition into Hamilton cycles and a perfect matching.

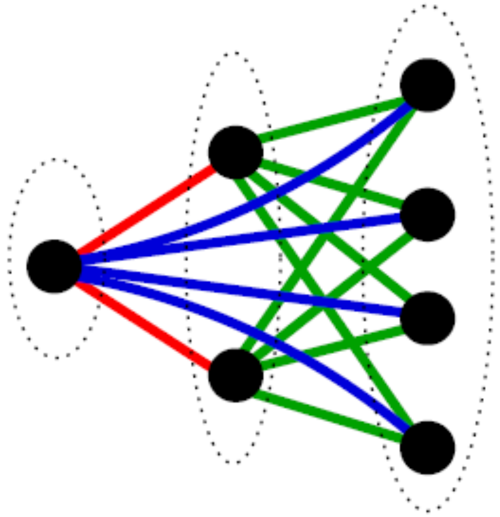




regular of
odd degree

Complete multipartite graphs

A graph is a **complete multipartite graph** if its vertices can be partitioned into parts such that two vertices are adjacent if and only if they are from different parts.



Theorem (Laskar and Auerbach 1976)
A complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.

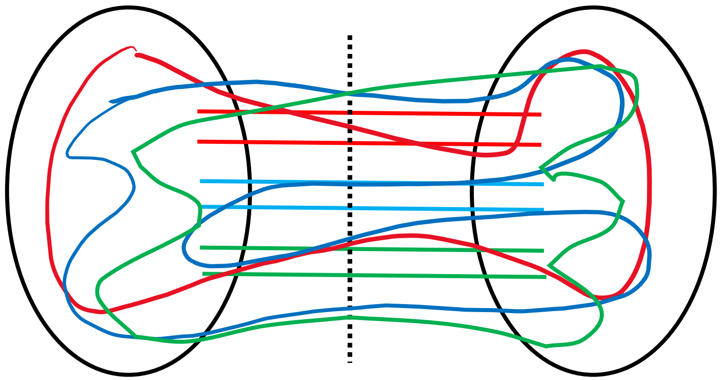
Theorem (Bryant, Hang, S.H. 2019)
A complete multipartite graph G with $n > 1$ vertices and m edges has a Hamilton **path** decomposition if and only if $t = \frac{m}{n-1}$ is an integer and $\Delta(G) \leq 2t$.

General context

Conjecture (Nash-Williams 1971, Jackson 1979)
Every connected k-regular graph of order at most $2k+1$ has a Hamilton decomposition.

✓ Proved for all sufficiently large k Csaba, Kühn, Lo, Osthus and Treglown [2015]

If a graph has a Hamilton decomposition with $\boldsymbol{t}$ **Hamilton cycles**, then it is $\mathbf{2}\boldsymbol{t}$**-edge connected.**

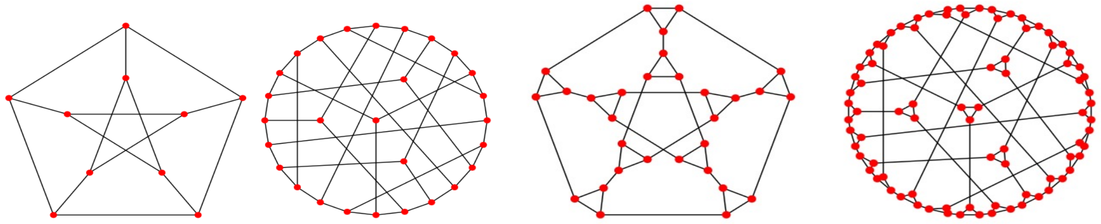


Theorem (Mader 1971)
Every connected k-regular vertex-transitive graph is k-edge-connected.

Vertex-transitive graphs

Does every connected vertex-transitive graph have a Hamilton decomposition?

Obvious 3-regular exceptions:



P C

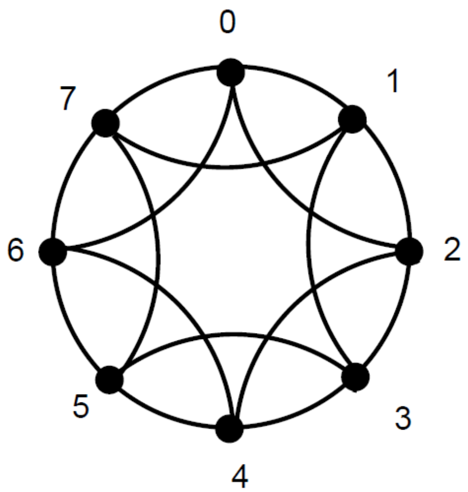
$L(P)$ and $L(C)$ are vertex-transitive, 4-regular and have no Hamilton decomposition.

Every other non-trivial connected vertex-transitive graph of order at most 31 has a Hamilton decomposition. Wagon [2014]

Theorem (Bryant and Dean, 2015)
There are infinitely many connected vertex-transitive graphs that have no Hamilton decomposition.

Cayley graphs

Cayley graphs



$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$

Alspach's Conjecture (1984)
Every connected $2k$-regular Cayley graph $\mathrm{Cay}(\mathrm{X}, S)$ on a finite abelian group has a Hamilton decomposition.

- ✓ 2-regular — the graph is a Hamilton cycle
- ✓ 4-regular — Bermond, Favaron, Maheo [1989]
- ✓ 6-regular — many partial results (Dean; Westlund)
- ✓ X odd order, S is a minimal generating set — Liu [1996]
- ✓ X even order, S is a *strongly* minimal generating set — Liu [2003]

Theorem (Bryant and Dean, 2015)
There exist connected $2k$-regular Cayley graphs on a finite **non-abelian** groups that have no Hamilton decomposition.

Cayley graphs

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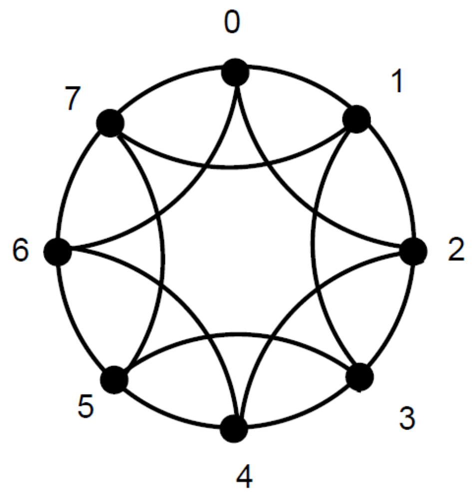
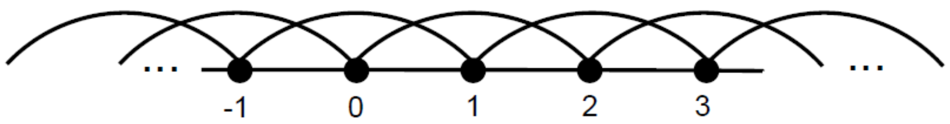


$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$

Question:
Does every connected $2k$-regular Cayley graph on an **infinite** abelian group have a Hamilton decomposition?



$\mathrm{Cay}(\mathbb{Z}, \pm\{1, 2\})$

Infinite Cayley graphs

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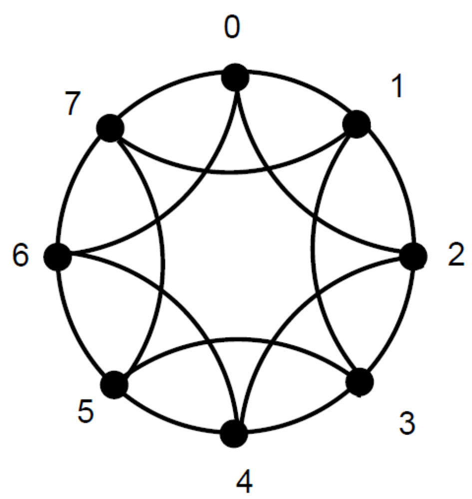

$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$

A Hamilton double-ray is a connected 2-regular spanning subgraph.

Question:
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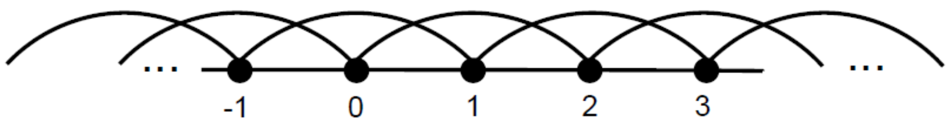

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Infinite Cayley graphs

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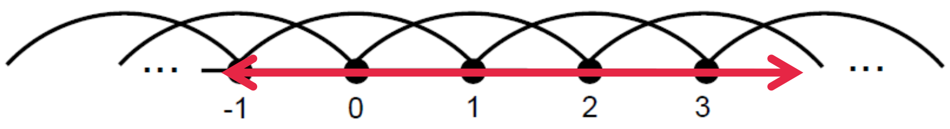


$\mathrm{Cay}(\mathbb{Z}, \pm\{1, 2\})$

A Hamilton double-ray is a connected 2-regular spanning subgraph.

A Hamilton decomposition is a decomposition into Hamilton double-rays.

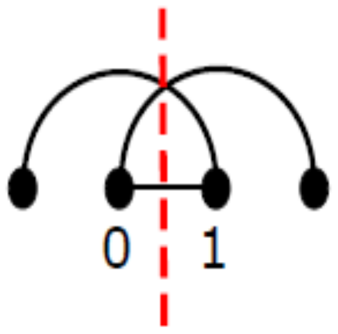
Theorem (Nash-Williams, 1959)
Every connected Cayley graph on a finitely-generated, infinite abelian group has a Hamilton double-ray.

$\Longrightarrow$ Every connected Cayley graph on a finitely-generated infinite abelian group with **infinite degree** has a Hamilton decomposition.

Infinite Cayley graphs of finite degree

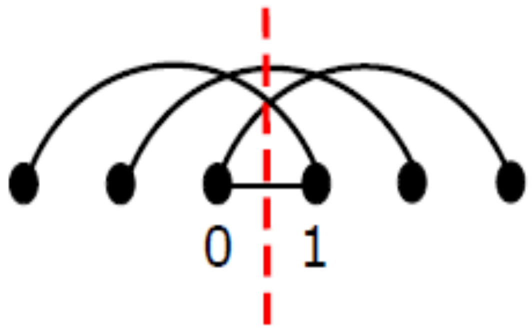
Necessary condition for a decomposition into k Hamilton double-rays:

$\mathrm{Cay}(\mathbb{Z}, \pm\{1, 2\})$



not admissible

$\mathrm{Cay}(\mathbb{Z}, \pm\{1, 3\})$



admissible

Each of the Hamilton double-rays uses an odd number of the edges that cross the dotted line.

$\Longrightarrow$ # edges crossing the dotted line $\equiv k \pmod 2$

Infinite analogue of Alspach's conjecture

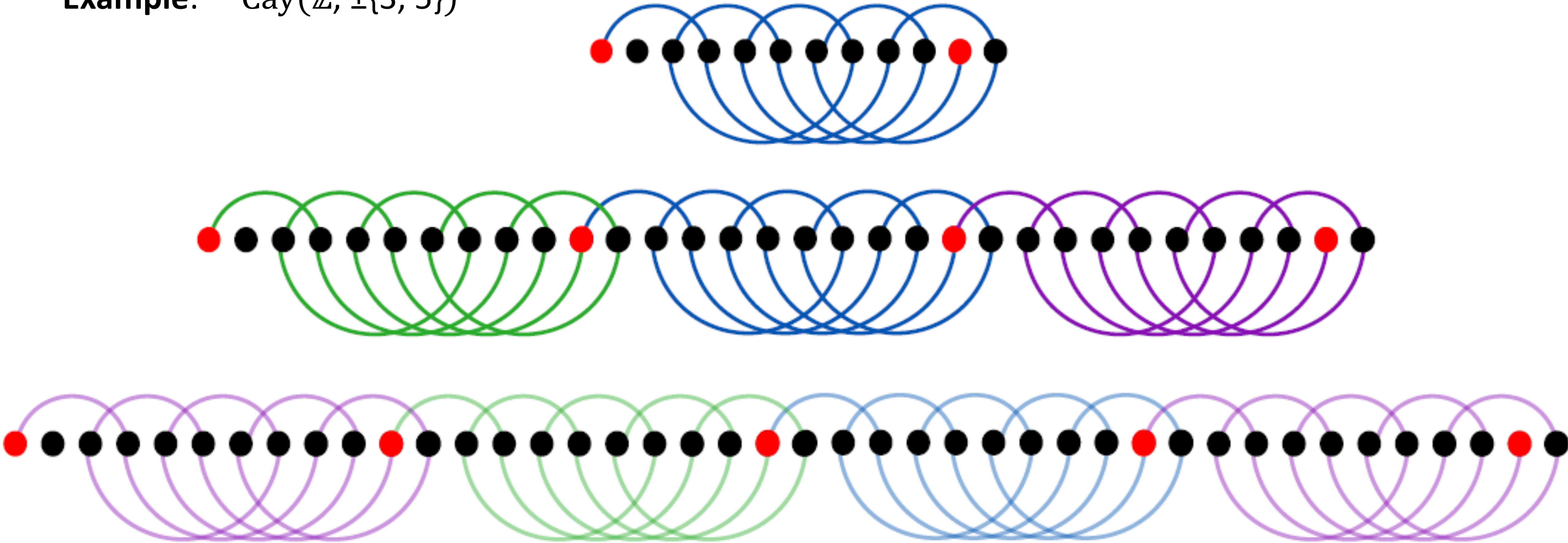
Open Question: Does every *admissible* connected $2k$-regular Cayley graph on an infinite abelian group have a Hamilton decomposition?

Cay(ℤ, S)

- ✓ S = $\{a, b\}$ — Bryant, S.H., Maenhaut, Webb [2017]
- ✓ S = $\{1,2, c\}$
- ✓ S = $\{1, 2, \ldots, k\}$
- ✓ S = $\{a_1, a_2, \ldots, a_{p-1}, p\}$ for $p \leq 23$ an odd prime, $p \nmid a_i$
- ✓ some other 6-regular cases — Gentle, Baldwin, Stephenson (unpublished)

Infinite analogue of Alspach’s conjecture

Example: $\mathrm{Cay}(\mathbb{Z}, \pm\{3, 5\})$



Infinite analogue of Alspach's conjecture

Open Question: Does every *admissible* connected $2k$-regular Cayley graph on an infinite abelian group have a Hamilton decomposition?

Cay($\mathbb{Z}$, S)

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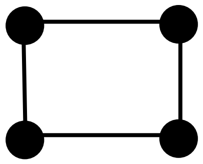
✓ **Cay($\mathbb{Z}^2$, S)** — Erde, Lehner, Pitz [2020]

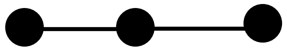
Theorem (Erde, Lehner, 2022): Every *admissible* connected 4-regular Cayley graph on an infinite abelian group has a Hamilton decomposition.

Graph products

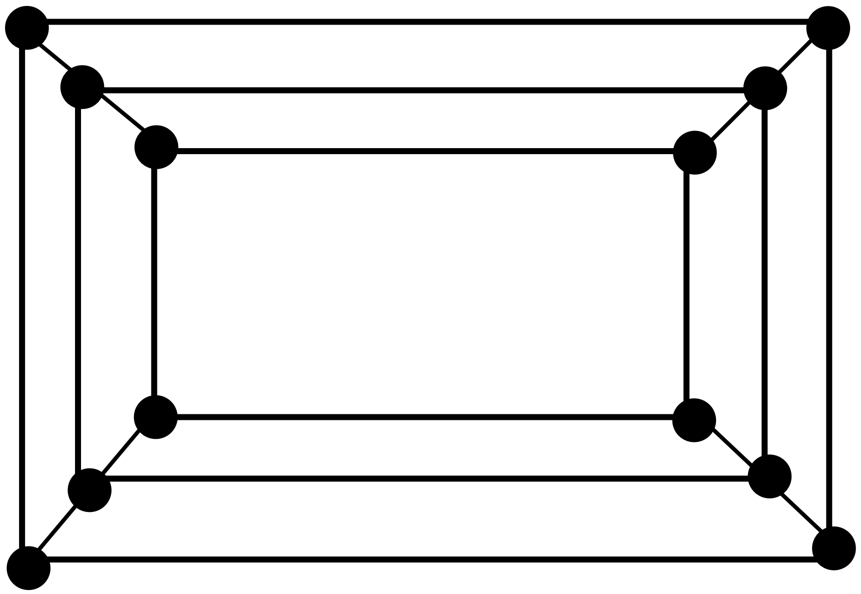
Cartesian product $G \times H$



C_4



P_3



$C_4 \times P_3$

Conjecture (Bermond 1978)
If G and H both have Hamilton decompositions, then $G \times H$ has a Hamilton decomposition.

Cartesian product

Conjecture (Bermond 1978)
If G and H both have Hamilton decompositions, then $G \times H$ has a Hamilton decomposition.

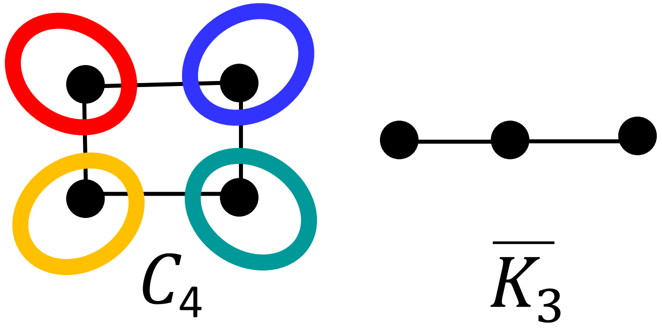
- ✓ $\mathrm{K}_n \times \mathrm{K}_n$ — Myers [1972]
- ✓ $\mathrm{K}_n \times \mathrm{K}_k$ — Aubert and Schneider [1981]
- ✓ $\mathrm{C}_n \times \mathrm{C}_k$ — Kotzig [1973]
- ✓ $\mathrm{C}_{n_1} \times \mathrm{C}_{n_2} \times \ldots \times \mathrm{C}_{n_r}$ — Alspach and Godsil [1985]

Theorem (Stong 1991)
If G and H have Hamilton decompositions into n and m Hamilton cycles, respectively, with $n \leq m$ then $G \times H$ has a Hamilton decomposition if one of the following holds:

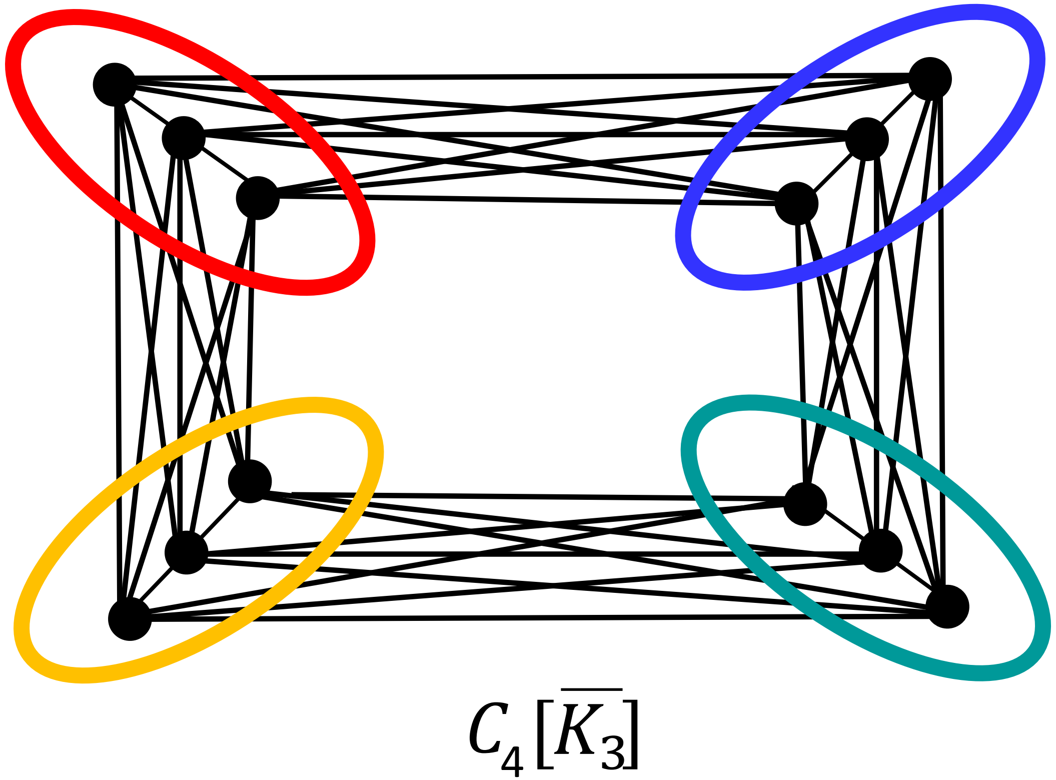
- $m \leq 3n$,
- $n \geq 3$,
- $|V(G)|$ is even,
- $|V(H)| \geq 6\left\lceil \frac{m}{n} \right\rceil - 3$

Wreath product* $G[H]$

*aka *lexicographic product* or *graph composition*



Replace every vertex u of G with a copy H_u of H, and for each edge of uv of G, join each vertex of H_u to each vertex of H_v.



Wreath product

For which G and H does $G[H]$ have a Hamilton decomposition?

- ✓ $\mathrm{K}_n[\mathrm{K}_k] \cong \mathrm{K}_{nk} \Leftrightarrow$ regular of even degree — Walecki [1892]
- ✓ $\mathrm{K}_n[\overline{\mathrm{K}_k}] \cong \mathrm{K}_{k,k,\dots,k} \Leftrightarrow$ regular of even degree — Laskar and Auerbach [1976]

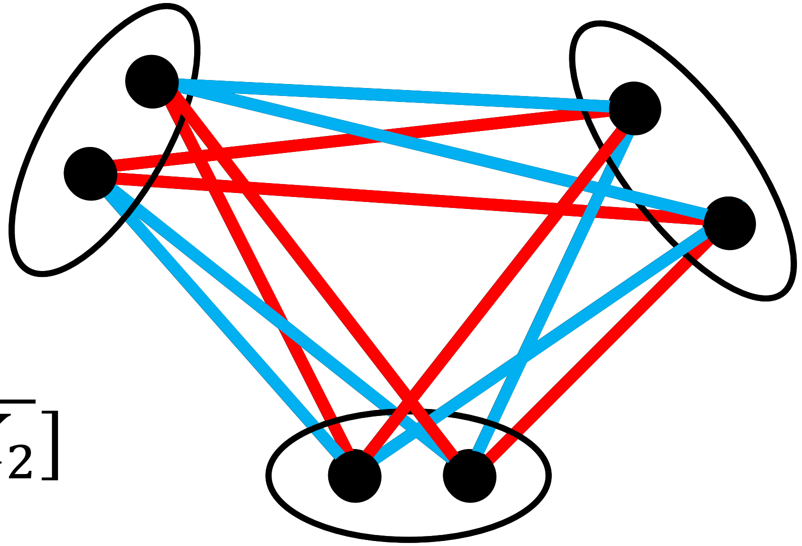
- ✓ $\mathrm{C}_n[\overline{\mathrm{K}_k}]$ — Bermond [1978], Laskar [1978]
- ✓ $\mathrm{C}_n[\mathrm{C}_k]$ where n is odd — Laskar [1978]

Theorem (Baranyai and Szasz 1981)
If G and H both have Hamilton decompositions, then $G[H]$ has a Hamilton decomposition.

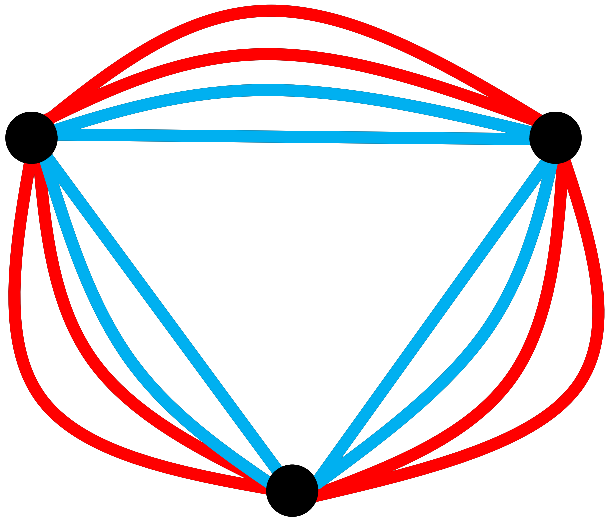
$$\left.\begin{matrix} G[\mathrm{K}_k] \\ \\ G[\overline{\mathrm{K}_k}] \end{matrix}\right\} \text{ Hamilton decomposition } \quad \Longrightarrow \; G \text{ is regular and connected}$$

Collapsed graph

$C_3[\overline{K_2}]$



$C_3[\overline{K_2}]^*$

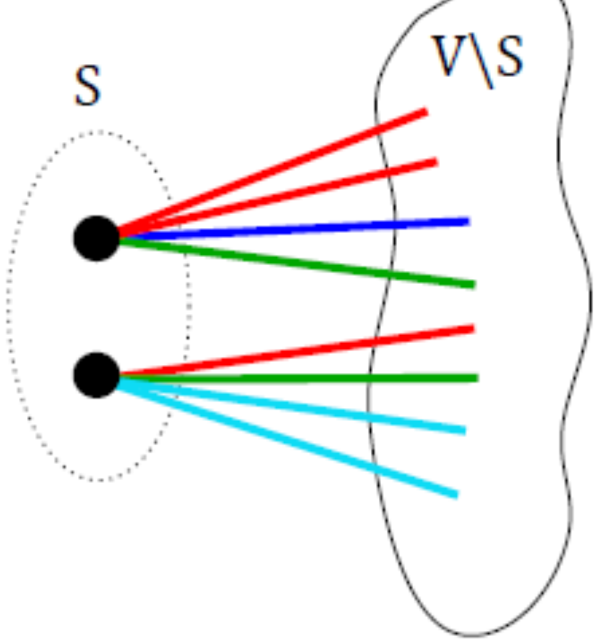


Decomposition into Hamilton cycles $\Longrightarrow$ $\Longleftarrow$ Connected 4-factorisation

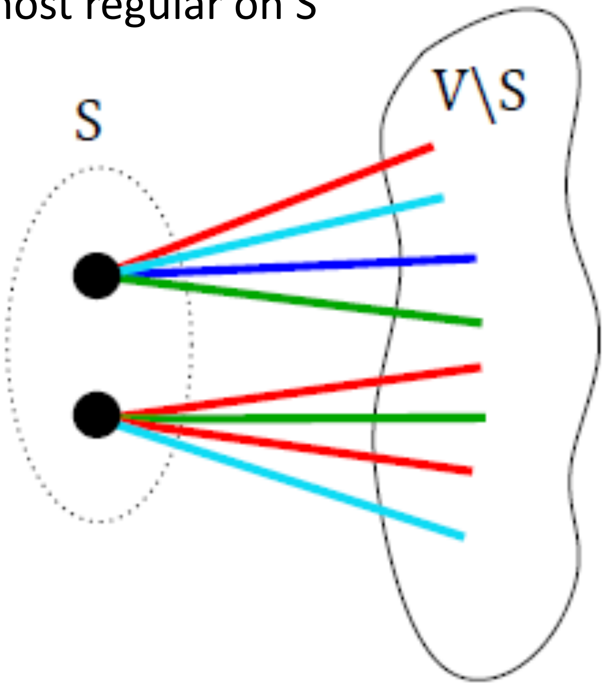
Lemma (Bryant, S.D., Hang 2023+)
Let G be a graph and let H be either K_k or $\overline{K_k}$. Then $G[\mathrm{H}]$ has a Hamilton decomposition if and only if $G[\mathrm{H}]^*$ has a connected $2k$–factorisation.

Almost regular edge colourings

not almost regular on S

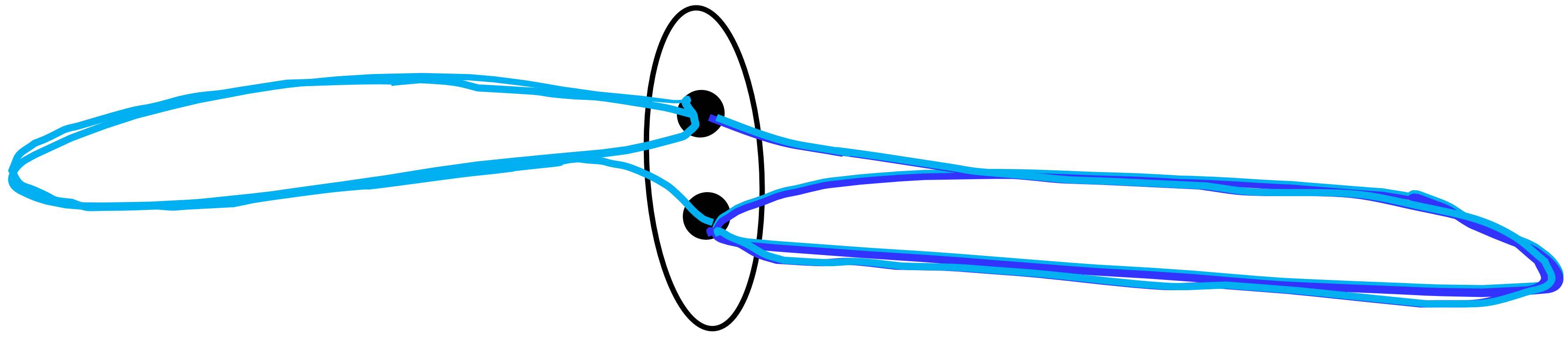


almost regular on S



Lemma (Bryant 2016)
If G is a graph with an edge colouring and $S \subseteq \mathrm{V}(G)$ such that any permutation of S is an automorphism of G, then there exists an edge colouring of G that has the “same properties” and is almost regular on S.

Making the 2-factors connected



Wreath product

Let G be a connected d-regular graph and let H be either K_k or $\overline{K_k}$ where $k \geq 2$.
Does $G[\mathrm{H}]$ have a Hamilton decomposition whenever it is $2t$-regular (and $2t$ -edge-connected)?

- ✓ $k \geq d$
- ✓ X = K_k and $k \geq \frac{d+2}{2}$
- ✓ d even, X = $\overline{\mathrm{K}_k}$ and $k \geq \frac{d}{2}$
- ✓ G has a 1-factorisation, X = $\overline{\mathrm{K}_k}$ and $k \geq \frac{d}{2}$
- ✓ other similar sufficient conditions

Bryant, S.D., Hang 2023+

✓ $d \leq 4$ except possibly $G[\overline{K_2}]$ when G is 3-regular, bridgeless, no 1-factorisation (*snark*)

? $5 \leq d \leq 7$

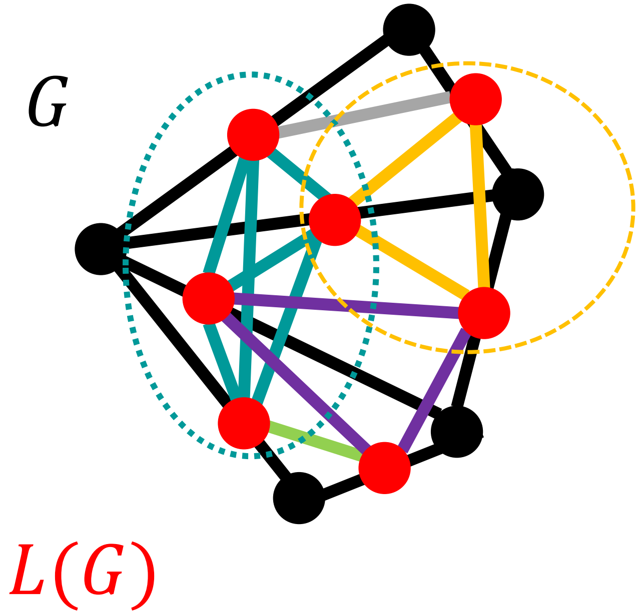
We checked well-known snarks and do not know of a counterexample

✗ $d \geq 8$, $d \equiv 0 \pmod 4$ construct $G[\overline{K_2}]$ $2d$-regular, $2d$-edge-connected but **non-Hamiltonian**

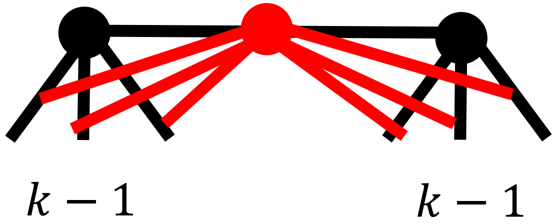
Line graphs

Line graphs

Given a graph G, the **line graph** of G, denoted $L(G)$, is the graph whose vertices are the edges of G and in which two vertices are adjacent if and only if the corresponding edges of G are adjacent.



If G is k-regular, then $L(G)$ is $(2k-2)$-regular.



Theorem (Kotzig 1964)
A 3-regular graph G is Hamiltonian if and only if $L(G)$ has a Hamilton decomposition.

Bermond's conjecture

Conjecture (Bermond 1988)
If G has a Hamilton decomposition, then is $L(G)$ has a Hamilton decomposition.

✓ 2-regular G	$L(G)$ is a cycle	
✓ 3-regular G	$L(G)$ is 4-regular	Kotzig [1964]
✓ 4-regular G	$L(G)$ is 6-regular	Jaeger [1983]
✓ 5-regular G	$L(G)$ is 8-regular	Pike [1995]
✓ k-regular bipartite G with k odd		Pike [1995]
✓ k-regular G with $k \equiv 0 (\mathrm{mod}\ 4)$		Muthasamy and Paulraja [1995]

Theorem (Bryant, Maenhaut, Smith 2015+ *)
If G has a Hamilton decomposition, then is $L(G)$ has a Hamilton decomposition.

*Ben Smith presented a proof of Bermond's conjecture at 39ACCMCC in Brisbane, 2015.

Strengthening Bermond's conjecture

Theorem (Kotzig 1964)
A 3-regular graph G is Hamiltonian if and only if $L(G)$ has a Hamilton decomposition.

$$G \text{ is Hamiltonian} \overset{?}{\Longrightarrow} L(G) \text{ has a Hamilton decomposition}$$

Theorem (Muthasamy and Paulraja, 1995, and Zahn 1992)
If G is k-regular and Hamiltonian (for k even), then $L(G)$ can be decomposed into Hamilton cycles and a 2-factor.

Theorem (Bryant, S.H., Maenhaut, Smith 2020+ *)
If G is k-regular and Hamiltonian (for k even) or contains a Hamiltonian 3-factor (for k odd) then $L(G)$ has a Hamilton decomposition.

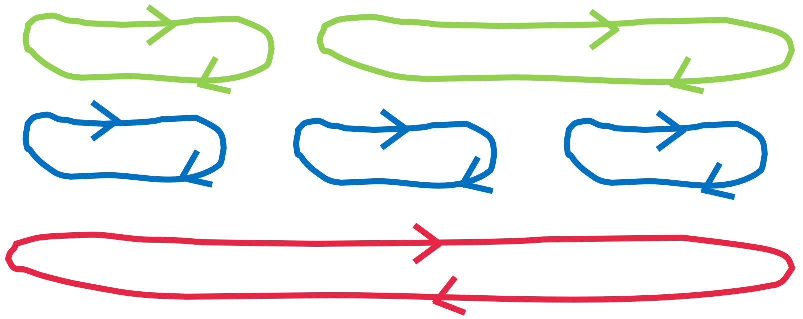
***Paper is on arXiv**

Strengthening Bermond's conjecture

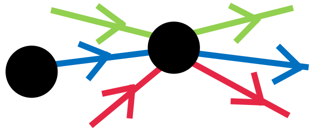
Theorem (Bryant, S.H., Maenhaut, Smith 2020+)
If G is k-regular and Hamiltonian (for k even) or contains a Hamiltonian 3-factor (for k odd) then $L(G)$ has a Hamilton decomposition.

Proof idea:

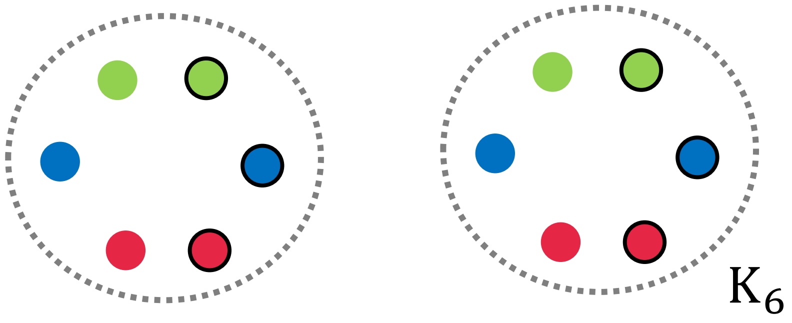
6-regular G
Hamiltonian



Orient the 2-factors



10-regular $\mathrm{L}(G)$



Strengthening Bermond's conjecture

Theorem (Bryant, S.H., Maenhaut, Smith 2020+)
If G is k-regular and Hamiltonian (for k even) or contains a Hamiltonian 3-factor (for k odd) then $L(G)$ has a Hamilton decomposition.

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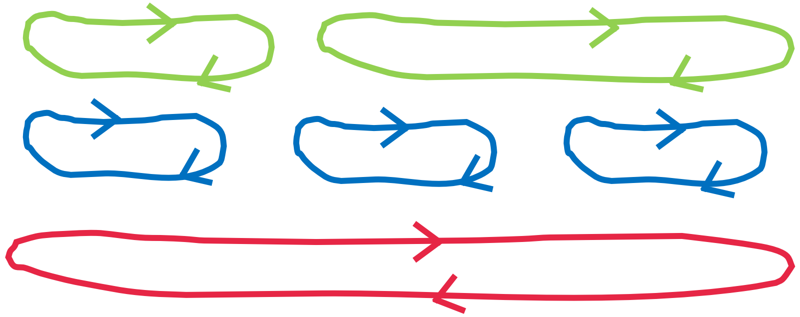
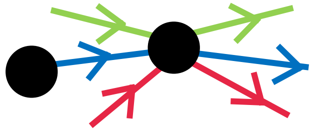
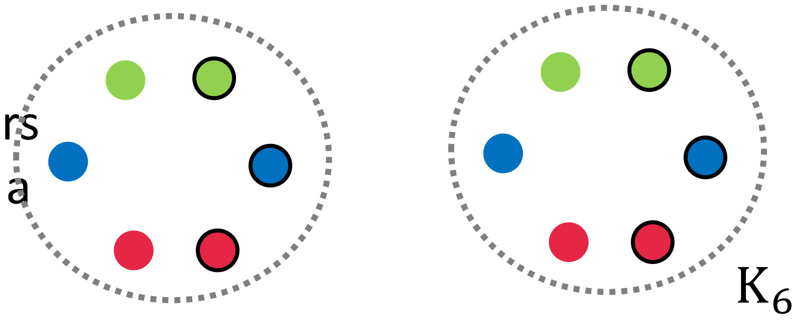


Orient the 2-factors

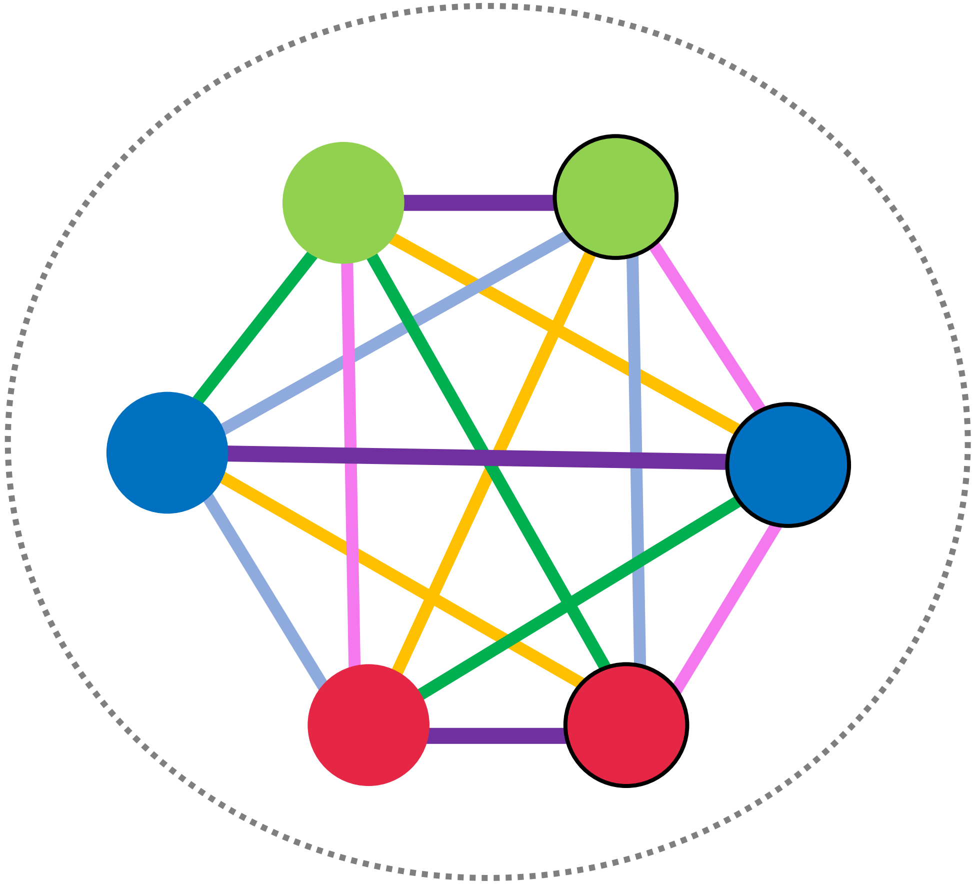


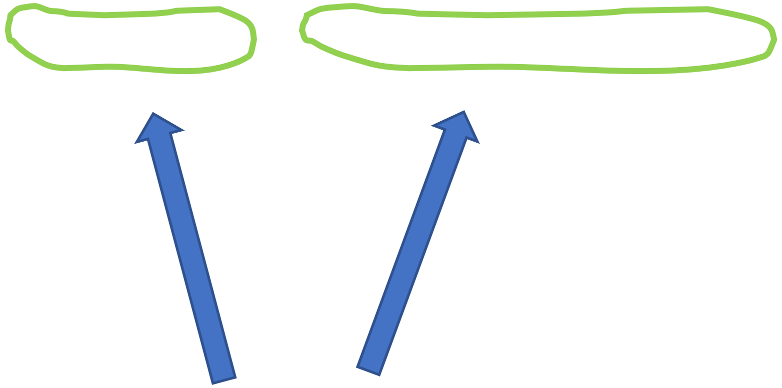
10-regular $\mathrm{L}(G)$

Colour edges of the K_6 with 5 colours so that putting them together gives a Hamilton cycle in each colour



Hamilton fragments



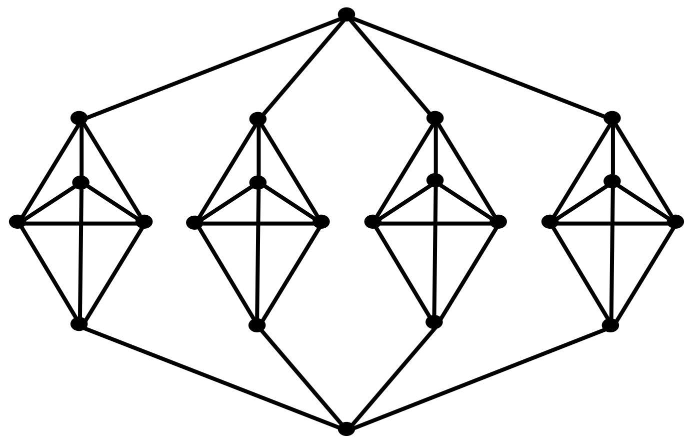


At least one vertex in each component gets an “alternate” Hamilton fragment

Hamiltonicity of G not necessary

Theorem (Bryant, Maenhaut, Smith, 2018)
For each integer $k \geq 4$ there exists a k-regular **non-Hamiltonian** graph G such that $L(G)$ has a Hamilton decomposition.

Example:



Theorem (Jackson, 1991)
If G is a 3-connected 4-regular graph, then $L(G)$ has a Hamilton decomposition.

Conjectured for 3-connected $2k$-regular

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Thanks for listening!

Summary of some open problems

1. Prove (or disprove) Alspach's conjecture for $k \geq 3$ that every connected $2k$-regular Cayley graph on a finite abelian group has a Hamilton decomposition.

2. For $k \geq 3$, characterise the connected $2k$-regular Cayley graphs on infinite abelian groups that have a decomposition into Hamilton double-rays.

3. For every snark G, does $G[\overline{K_2}]$ have a Hamilton decomposition?

4. For $k \geq 4$, characterise the k-regular graphs whose line graph has a Hamilton decomposition.

Regular highly connected graphs

Conjecture (Häggkvist 1976, Bollobás 1978)
Every t-connected k-regular graph of order at most $(t+1)k$ is Hamiltonian.

✓ $t = 2$ Jackson [1980]

✓ $t = 3$ when n is sufficiently large Kühn, Lo, Osthus, Staden [2016]

✓ ✗ Counterexamples for all $t \geq 4$ Jung [1984] and Jackson, Li and Zhu [1991]

Conjecture (Häggkvist 1976)
Every 2-connected k-regular <u>bipartite</u> graph of order at most $6k$ is Hamiltonian.

✓ for orders $\leq 6k - 38$ Jackson [1994]