A graph co-spectral to $\mathrm{NO}^{+}(8,2)$

joint work with S. Adriaensen, R. Bailey, M. Rodgers

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Thank you Cheryl, John, Alice, Gordon, Michael, Luke, Anton

The $\mathrm{NO}^+(8,2)$-graph

Let Q be a quadratic form on $\mathbb{F}_q^n$. Define $f : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q$ as

$$f(v,w) = Q(v+w) - Q(v) - Q(w)\,.$$

Then f is the *polar form* of Q.

The $\mathrm{NO}^+(8,2)$-graph

Let Q be the quadratic form $x_0x_1 + x_2x_3 + x_4x_5 + x_6x_7$ on $\mathbb{F}_2^8$, and f its polar form. Define

- vertices as the non-zero vectors of $\mathbb{F}_2^8$, which are non-singular with relation to Q,
- $x \sim y \iff f(x,y) = 0$.

The $\mathrm{NO}^+(8,2)$-graph

- This is an $\mathrm{srg}(120, 63, 30, 36)$, with automorphism group $\mathrm{P\Gamma O}^+(8,2)$.
- One example of a large class of graphs, called *Fischer graphs*[1]

[1] Brouwer-Van Maldeghem, chapter 5

Orbital graphs

- Let G be a group acting transitively on a set Ω.
- The *orbitals* of G are its orbits on $\Omega \times \Omega$. The *rank* of G is the number of orbitals.
- Each orbital o defines a directed graph with vertex set Ω and $v \to w \iff (v, w) \in o$.

Orbital graphs

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A *graph has rank r* if $\mathrm{Aut}(G)$ has r orbitals on its vertices.

Question of Robert Bailey

There exists an $\mathrm{srg}(120, 63, 30, 36)$ arising from a rank-7 action of $\mathrm{Sym}(7)$ (Brouwer/Van Maldeghem). Can we find a geometrical description?

Question of Robert Bailey

v	k	λ	μ	#	rk	G	suborbit sizes	ref
63	30	13	15	1	4	$\mathrm{PSU}_3(3).2$	$1, 6, 24, 32$	§10.22
81	30	9	12	1	4	$3^4 : (2 \times \mathrm{S}_6)$	$1, 20, 30, 30$	§10.29
105	32	4	12	1	4	$\mathrm{L}_3(4).D_{12}$	$1, 8, 32, 64$	§10.33
120	42	8	18	1	4	$\mathrm{L}_3(4).2^2$	$1, 21, 42, 56$	§10.37
120	56	28	24	1	4	S_{10}	$1, 21, 35, 63$	p. 299
120	56	28	24	1	7	S_7	$1, 7, 14, 14, 21, 21, 42$	
144	39	6	12	1	6	$\mathrm{L}_3(3).2$	$1, 13, 26, 26, 39, 39$	§10.45
144	55	22	20	1	4	$\mathrm{M}_{12}.2$	$1, 22, 55, 66$	§10.46
144	66	30	30	2	4	$\mathrm{M}_{12}.2$	$1, 22, 55, 66$	§10.46
175	72	20	36	1	4	$\mathrm{P\Sigma U}_3(5)$	$1, 12, 72, 90$	p. 269
208	75	30	25	1	4	$\mathrm{P\Gamma U}_3(4)$	$1, 12, 75, 120$	$NU_3(4)$
231	30	9	3	1	4	$M_{22}.2$	$1, 30, 40, 160$	§10.54

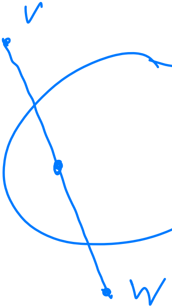
continued...

A geometric description?

Joint work with: Sam Adriaensen, Robert Bailey, Morgan Rodgers.

- Set up $\mathrm{NO}^+(8,2)$ in a geometric way.

$Q^+(7,2)$



A geometric description?

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- Set up $\mathrm{NO}^+(8,2)$ in a geometric way.
- Find copies of S_7 in $\mathrm{Aut}(\mathrm{NO}^+(8,2))$.

A geometric description?

Joint work with: Sam Adriaensen, Robert Bailey, Morgan Rodgers.

- Set up $\mathrm{NO}^+(8,2)$ in a geometric way.
- Find copies of S_7 in $\mathrm{Aut}(\mathrm{NO}^+(8,2))$.
- Look at orbitals of S_7 on the vertices.
- Try out combinations of orbitals to see if we find an srg with the same parameters.
- Look for a geometrical description by exploring its adjacency relation and looking at how the group acts on the other objects.

The hyperbolic quadric $Q^+(7, q)$

$Q^+(7,2): \quad X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0$

The hyperbolic quadric $\mathrm{Q}^+(7, q)$

- The geometry of totally singular subspaces of $\mathbb{F}_2^8$ of dimension at least 1, with respect to a quadratic form of hyperbolic type.
- This is a *finite classical polar space* of rank 4, embedded in a 7-dimensional projective space …

The hyperbolic quadric $\mathrm{Q}^+(7, q)$

- The geometry of totally singular subspaces of $\mathbb{F}_2^8$ of dimension at least 1, with respect to a quadratic form of hyperbolic type.
- This is a *finite classical polar space* of rank 4, embedded in a 7-dimensional projective space ...
- i.e. it contains points, lines, planes, solids.

Ovoids and spreads of $\mathrm{Q}^+(7,q)$

Key observation

One orbit of S_7 on the generators consists of 7 mutually skew generators, another orbit a pair of mutually skew generators, together making a *spread* of $\mathrm{Q}^+(7,q)$.

Ovoids and spreads of $Q^+(7, q)$

Let $\mathcal{P}$ be a finite classical polar space.

- An ovoid of $\mathcal{P}$ is a set $\mathcal{O}$ of points such that every generator meets $\mathcal{O}$ in exactly one point.
- A spread of $\mathcal{P}$ is a set $\mathcal{S}$ of generators of $\mathcal{P}$ such that every point is contained in exactly one element of $\mathcal{S}$.

Ovoids and spreads of $\mathrm{Q}^+(7, q)$

Let $\mathcal{P}$ be a finite classical polar space.

- An ovoid of $\mathcal{P}$ is a set $\mathcal{O}$ of points such that every generator meets $\mathcal{O}$ in exactly one point. This is a coclique in the collinearity graph of $\mathcal{P}$, of largest possible size.
- A spread of $\mathcal{P}$ is a set $\mathcal{S}$ of generators of $\mathcal{P}$ such that every point is contained in exactly one element of $\mathcal{S}$. This is a clique in the opposition graph on the generators of $\mathcal{P}$, of largest possible size.

Ovoids and spreads of $\mathrm{Q}^+(7,q)$

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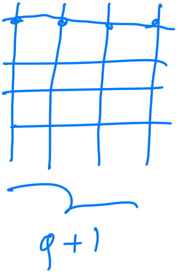
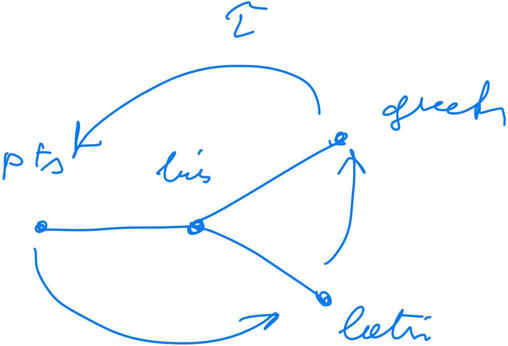
Note

The generators of $\mathrm{Q}^+(7,q)$ come in two systems (*greek* and *latin* generators). Generators belonging to one system meet in projective dimension -1, 1, or 3. Hence a spread consists of generators all belonging to one of the systems.

Ovoids and spreads of $Q^+(7, q)$

$Q^+(3, q)$ $\quad$ $Q^+(5, q)$ $\quad$ $Q^+(7, q)$





Ovoids and spreads of $\mathrm{Q}^+(7, q)$

The quadric $\mathrm{Q}^+(7, q)$ allows a *triality*, i.e. a map or order 3, preserving incidence, and mapping

- lines on lines,
- points on greeks,
- greeks on latins,
- latins on points.

Hence a triality maps a spread of latins on an ovoid, and an ovoid on a spread of latins.

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Corollary

Ovoids and spreads of $Q^+(7, q)$ are equivalent objects.

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Corollary

Ovoids and spreads of $Q^+(7, q)$ are equivalent objects.

Existence or non-existence of ovoids (and hence spreads) is not settled for $Q^+(7, q)$ for all values of q.

Ovoids and spreads of $\mathrm{Q}^+(7, q)$

1205

p.j. cameron – c.e. praeger

PARTITIONING INTO STEINER SYSTEMS

An *overlarge set of Steiner systems* $S(k-1,k,n)$ is a partition of the set of all k-subsets of an $(n+1)$-set into such systems (each omitting one point). Breach and Street showed that there are just two such sets up to isomorphism for $k = 4$, $n = 8$ both admitting 2-transitive groups. Our purpose here is twofold:

(i) to give a short proof of this result, using the geometry of the $0^+(8,2)$ quadric (including triality);
(ii) to show the non-existence of overlarge sets of $S(5,6,12)$s.

1 Preliminaries

Let $P(k,n)$ denote an overlarge set of $S(k-1,k,n)$ systems, that is, a partition of the set of k-subsets of an $(n+1)$-set into Steiner systems $S(k-1,k,n)$.

Proposition (1) –

(i) $P(k,n)$ consists of $n+1$ Steiner systems, and each point is omitted by one of them.
(ii) If $P(k,n)$ exists, then $P(k-1,n-1)$ exists.
(iii) If $S(k,k+1,n+1)$ exists, then $P(k,n)$ exists.

Proof — The number of Steiner systems is

$$\binom{n+1}{k} / |S(k-1,k,n)| = n+1 .$$

We form the derived system of a $P(k,n)$ with respect to a point p, by taking the derived systems with respect to p of all the Steiner systems involving p; it is easily checked that this is a $P(k-1,n-1)$. Hence by the first sentence (with $n-1$ replacing n), p lies in n of the Steiner systems, and is omitted

61

Ovoids and spreads of $Q^+(7, q)$

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(i) to give a short proof of this result, using the geometry of the $O^+(8,2)$ quadric (including triality);

(ii) to show the non-existence of overlarge sets of $S(5,6,12)$s.

Ovoids and spreads of $\mathrm{Q}^+(7, q)$

which maps lines to lines and preserves incidence. If p and q are points, then $p\tau \cap q\tau = \emptyset$ if and only if p and q are not perpendicular. So $Q\tau$ is a set of 9 pairwise disjoint solids, that is, a *spread* of solids. Every spread arises as the image of an ovoid under τ or τ^2. Thus the stabiliser of a spread is A_9.

More fun with spreads and ovoids

From Cameron-Praeger (1991), we know

- There is a unique spread, with automorphism group A_9.
- There are 960 copies of the unique ovoid.
- There are 2 orbits of ovoids under A_9:
  - one orbit O_1 of length 120, each ovoid having stabilizer $\mathrm{P\Gamma L}(2, 8)$
  - one orbit O_2 of length 840, each ovoid having stabilizer $\mathrm{ASL}(2, 3)$

The construction

- Fix the spread $\mathcal{S}$. Choose two solids π_1, π_2. The setwise stabilizer of $\{\pi_1, \pi_2\}$ in $\mathrm{Aut}(\mathcal{S})$ is S_7.
  - Any point $p \in \mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)$ determines a unique point $p_1 \in \pi_1$ and $p_2 \in \pi_2$ and vice versa.
  - Given two points $p_1 \in \pi_1$ and $p_2 \in \pi_2$, there is a unique ovoid $\mathcal{O} \in O_1$ meeting π_1 in p_1 and π_2 in p_2.
  - S_7 acts transitively on the points of $\mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)$.

The construction

- A vertex v determines a unique ovoid $\mathcal{O} \in O_1$.
- The hyperplane $v^\perp$ meets $\mathrm{Q}^+(7,2)$ in a parabolic quadric $\mathrm{Q}(6,2)$, meeting $\mathcal{O}$ in a *maximal partial ovoid* $\mathcal{O}'$ of size 7.
- Each set $\mathcal{O}'' \subset \mathcal{O}'$ with $|\mathcal{O}''| = 6$ will span a 5-dimensional space Π_5, and will be a *maximal partial ovoid* of the elliptic quadric $\mathrm{Q}^-(5,2) = \Pi_5 \cap \mathrm{Q}^+(7,2)$.
- The line $\Pi_5^\perp$ will contain no points of $\mathrm{Q}^+(7,2)$.

The construction

Let v, w be two different vertices. (Recall: v determines $\mathcal{O}'$ uniquely).
Then $v \sim w$ if

(a) $\langle v, w \rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\pi_1 \cup \pi_2$; or

(b) $\langle v, w \rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\mathcal{O}'$; or

(c) $\langle v, w \rangle$ is a line skew to $\mathrm{Q}^+(7,2)$ and $\langle v, w \rangle^\perp$ does not meet $\mathcal{O}'$ in 6 points.

The construction

Let v, w be two different vertices. (Recall: v determines $\mathcal{O}'$ uniquely).
Then $v \sim w$ if

(a) $\langle v, w\rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\pi_1 \cup \pi_2$;
This gives 14 adjacencies;

(b) $\langle v, w\rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\mathcal{O}'$;
This gives 7 adjacencies;

(c) $\langle v, w\rangle$ is a line skew to $\mathrm{Q}^+(7,2)$ and $\langle v, w\rangle^\perp$ does **not** meet $\mathcal{O}'$ in 6 points.
There are 28 lines on v skew to $\mathrm{Q}^+(7,2)$. Because there are exactly 7 sets $\mathcal{O}''$ of size 6, there are exactly 7 lines l_i on v skew to $\mathrm{Q}^+(7,2)$ such that $l_i^\perp$ meets $\mathcal{O}'$ in such a set $\mathcal{O}''$. So there are exactly 21 skew lines satisfying the condition, each line contains 2 vertices adjacent to v, hence 42 adjacencies.

The construction

To do (without computer)

- Show that λ and μ have the desired values.
- Show that it is indeed a rank 7 graph.
- Investigate (co)-cliques of this graph.

One more thing ...

One more thing . . .

One more thing . . .

One more thing ...

- Each year: call for postdoctoral fellowships from the Research Foundation (FWO).
- 3 years, junior and/or senior
- Starting date: 1st of October or November
- Call opens early September, deadline for submission is December 1st.
- Results are known end of May, early June.
- Interested? Jan.De.Beule@vub.be

```
LoadPackage("fining");

# The NO+(8,2) graph.
q := 2;
ps := HyperbolicQuadric(7,q);
pg := PG(7,q);
pts := AsList(Points(pg));
vertices := Filtered(pts,x->not x in ps);
aut := CollineationGroup(ps);
delta := PolarityOfProjectiveSpace(ps);
adj := function(x,y)
if x = y then return false;
else return x in y^delta;
fi;
end;
graph := Graph(aut,vertices,OnProjSubspaces,adj,true);
GlobalParameters(graph);
group := AutomorphismGroup(graph);
StructureDescription(group);
StructureDescription(aut);

#The S7 graph

#First the spread of Q+(7,2) and the orbit of ovoids under the stabilizer of
 the spread
#Note: we'll find the spread as coclique in the opposition graph on the
 generators.
adj4 := function(x,y)
if x=y then
return false;
else return ProjectiveDimension(Meet(x,y)) = -1;
fi;
end;

solids := AsList(Solids(ps));;

oppgraph := Graph(aut,solids,OnProjSubspaces,adj4,true);
spreads := CompleteSubgraphsOfGivenSize(oppgraph,9,2,true);
spread := spreads[1];
spreadsolids := solids{spread};

#Similarly: an ovoid
adj3 := function(x,y)
if x=y then
return false;
else return not IsCollinear(ps,x,y);
fi;
end;

ccollgraph := Graph(aut,Set(Points(ps)),OnProjSubspaces,adj3,true);
ovoids := CompleteSubgraphsOfGivenSize(ccollgraph,9,2,true);
ovoid := ovoids[1];
ovoidpts := Set(Points(ps)){ovoid};

#Orbits on the ovoids
```

```
#All copies of the ovoid:
ovoidsorbit := FiningOrbit(aut,ovoidpts,OnSets); #960 ovoids.

#Stabilize the spread
stabspread := FiningSetwiseStabiliser(aut,Set(spreadsolids));
StructureDescription(stabspread);
stabovoid := FiningSetwiseStabiliser(aut,ovoidpts);
StructureDescription(stabovoid);
ovoidsorbit2 := FiningOrbits(stabspread,List(ovoidsorbit),OnSets);

shortorbit := First(ovoidsorbit2,x->Length(x)=120);
longorbit := First(ovoidsorbit2,x->Length(x)=840);

#Let's pick two solids of the spread and compute the pointwise/setwise
 stabiliser groups in stabspread.
s1 := spreadsolids[1];
s2 := spreadsolids[2];
stab := FiningSetwiseStabiliser(stabspread,[s1,s2]);
StructureDescription(stab); #This is S_7!

#given a copy of the ovoid, we would like to determine a vertex as follows:
 connect the two points of the ovoids in s1, respectively s2, this gives a
 hyperbolic line, having exactly one vertex.

vertex_from_ovoid := function(ovoid)
local p1,p2,line,points;
p1 := Filtered(ovoid,x->x in s1)[1];
p2 := Filtered(ovoid,x->x in s2)[1];
line := Span(p1,p2);
points := Difference(List(Points(line)),[p1,p2]);
return points[1];
end;

two_points_from_vertex := function(x)
local hyperplane,space,y,pts;
space := AmbientSpace(x);
hyperplane := First(Hyperplanes(space),y->not x in y);
pts :=
 Filtered(Points(Span(Meet(Meet(hyperplane,Span(x,s1)),Meet(hyperplane,Span(
 x,s2))),x)),y->y <> x);
return List(pts,y->Embed(ps,y));
end;

adj2 := function(x,y)
local t,ovoid,two;
if x = y then
    return false;
elif x in y^delta then
    t := First(Points(Span(x,y)),z->z <> x and z <> y);
    if t in s1 or t in s2 then
        return true;
    else
        two := two_points_from_vertex(x);
        ovoid := First(shortorbit,x->Intersection(two,x)=two);
        return t in ovoid;
```

```
    fi;
else
    two := two_points_from_vertex(x);
    ovoid := First(shortorbit,x->Intersection(two,x)=two);
    return Number(ovoid,z->z in Span(x,y)^delta) <> 6;
fi;
end;

s7graph := Graph(stab,vertices,OnProjSubspaces,adj2,true);
GlobalParameters(s7graph);
test := AutomorphismGroup(s7graph);
StructureDescription(test);

#check whether this is a rank 7 graph.

pairs := Tuples(vertices,2);;
action := function(pair,g)
return [pair[1]^g,pair[2]^g];
end;
orbitals := OrbitsDomain(stab,pairs,action);;
Length(orbitals);
```