Proper Minor-Closed Classes of Graphs have Assouad–Nagata Dimension 2

Marc Distel

School of Mathematics, Monash University

This talk is at the intersection of:

This talk is at the intersection of:

- Structural graph theory

This talk is at the intersection of:

- Structural graph theory
- Metric geometry

This talk is at the intersection of:

- Structural graph theory
- Metric geometry

In particular: Treating graphs as metric spaces, and using structural properties to solve metric problems.

Graph minors

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

A class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

A class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$, and is *proper* if $\mathcal{G}$ is not the class of all graphs.

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

A class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$, and is *proper* if $\mathcal{G}$ is not the class of all graphs.

Some easy examples:

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

A class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$, and is *proper* if $\mathcal{G}$ is not the class of all graphs.

Some easy examples:

- Planar graphs

Graph minors

A graph H is a *minor* of a graph G if H can be obtained from G with the following operations:

- Edge deletions
- Vertex deletions
- Contractions

A class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$, and is *proper* if $\mathcal{G}$ is not the class of all graphs.

Some easy examples:

- Planar graphs
- Linklessly embeddable graphs

Asymptotic and Assouad–Nagata dimension

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n
- Planar graphs have asymptotic and Assouad–Nagata dimension 2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n
- Planar graphs have asymptotic and Assouad–Nagata dimension 2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Applications:

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n
- Planar graphs have asymptotic and Assouad–Nagata dimension 2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Applications:
  - Geometric group theory

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n
- Planar graphs have asymptotic and Assouad–Nagata dimension 2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Applications:
  - Geometric group theory
  - Embeddings into Banach spaces

Asymptotic and Assouad–Nagata dimension

- Concepts from metric geometry
- Measures of the large-scale shape of a metric space
- $\mathbb{R}^n$ has asymptotic and Assouad–Nagata dimension n
- Planar graphs have asymptotic and Assouad–Nagata dimension 2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Applications:
  - Geometric group theory
  - Embeddings into Banach spaces
  - Travelling salesman problem (Erschler and Mitrofanov [2021]) (Assouad–Nagata dimension only)

Asymptotic dimension

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n + 1$ colours, such that the following rules hold:

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n+1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G

Asymptotic dimension

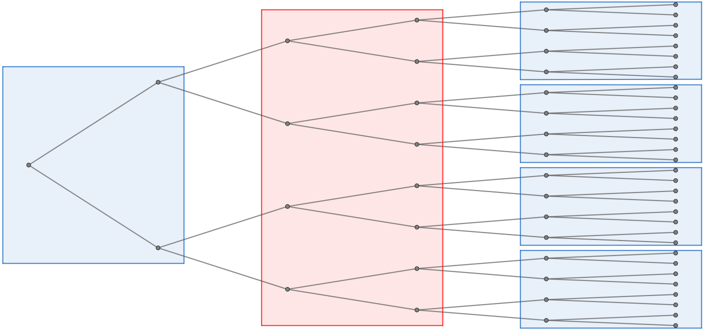
For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n+1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G
- Two vertices in the same part are at distance at most $f(r)$ in G

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n + 1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G
- Two vertices in the same part are at distance at most $f(r)$ in G



Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n+1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G
- Two vertices in the same part are at distance at most $f(r)$ in G

We then say that f is an *n-dimensional control function* for G.

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n+1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G
- Two vertices in the same part are at distance at most $f(r)$ in G

We then say that f is an *n-dimensional control function* for G.

If f is an n-dimensional control function for every graph G in a class $\mathcal{G}$, we say that f is an *n-dimensional control function* for $\mathcal{G}$.

Asymptotic dimension

For a graph G, given some integer $n \geqslant 0$, we want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every real number $r > 0$, we can find a partition $\mathcal{P}$, and give each part of one $n + 1$ colours, such that the following rules hold:

- Two part of the same colour are at distance more than r in G
- Two vertices in the same part are at distance at most $f(r)$ in G

We then say that f is an *n-dimensional control function* for G.

If f is an n-dimensional control function for every graph G in a class $\mathcal{G}$, we say that f is an *n-dimensional control function* for $\mathcal{G}$.

The *asymptotic dimension* of a class of graphs $\mathcal{G}$ is the smallest n such that $\mathcal{G}$ admits an n-dimensional control function, or ∞ otherwise (Gromov [1993]).

A problem

A problem

The control function $f(r)$ can be very big.

A problem

The control function $f(r)$ can be (and sometimes must be) very big.

A problem

The control function $f(r)$ can be (and sometimes must be) very big.

So let's consider when we can force it to be small.

A problem

The control function $f(r)$ can be (and sometimes must be) very big.

So let's consider when we can force it to be small.

A *dilation* is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \leqslant cr$ for some real number c.

A problem

The control function $f(r)$ can be (and sometimes must be) very big.

So let's consider when we can force it to be small.

A *dilation* is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \leqslant cr$ for some real number c.

The *Assouad–Nagata dimension* of a class of graphs $\mathcal{G}$ is the smallest n such that $\mathcal{G}$ admits an n-dimensional control function that is also a dilation, or ∞ otherwise (Assouad [1982], Nagata [1958]).

A problem

The control function $f(r)$ can be (and sometimes must be) very big.

So let's consider when we can force it to be small.

A *dilation* is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \leqslant cr$ for some real number c.

The *Assouad–Nagata dimension* of a class of graphs $\mathcal{G}$ is the smallest n such that $\mathcal{G}$ admits an n-dimensional control function that is also a dilation, or ∞ otherwise (Assouad [1982], Nagata [1958]).

From the definition, the Assouad–Nagata dimension is at most asymptotic dimension.

Examples

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])

Examples

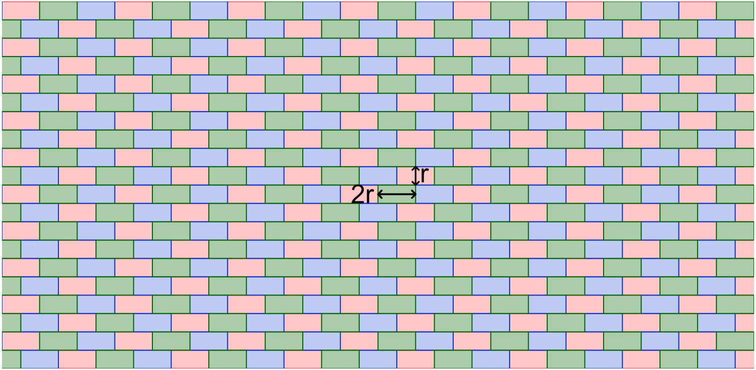
Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])



Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])
- Complete graphs (0)

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])
- Complete graphs (0)
- Planar graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])
- Complete graphs (0)
- Planar graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Bounded genus graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])
- Complete graphs (0)
- Planar graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Bounded genus graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

The asymptotic dimension and Assouad–Nagata dimension can also be arbitrarily far apart.

Examples

Classes for which asymptotic dimension and Assouad–Nagata dimension are the same:

- Paths and trees (1) (Bell and Dranishnikov [2008])
- d-dimensional grids (d) (Gromov [1993])
- Complete graphs (0)
- Planar graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])
- Bounded genus graphs (2) (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

The asymptotic dimension and Assouad–Nagata dimension can also be arbitrarily far apart. For example, 1-planar graphs have asymptotic dimension 2 but Assouad–Nagata dimension ∞.

The Dimension of Minor-Closed Classes

The Dimension of Minor-Closed Classes

Theorem (Ostrovskii and Rosenthal [2015])

If a proper minor-closed class excludes K_t, its Assouad–Nagata dimension is at most $4^t - 1$.

The Dimension of Minor-Closed Classes

Theorem (Ostrovskii and Rosenthal [2015])

If a proper minor-closed class excludes K_t, its Assouad–Nagata dimension is at most $4^t - 1$.

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Every proper minor-closed class has asymptotic dimension at most 2.

The Dimension of Minor-Closed Classes

Theorem (Ostrovskii and Rosenthal [2015])

If a proper minor-closed class excludes K_t, its Assouad–Nagata dimension is at most $4^t - 1$.

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Every proper minor-closed class has asymptotic dimension at most 2.

My contribution:

The Dimension of Minor-Closed Classes

Theorem (Ostrovskii and Rosenthal [2015])

If a proper minor-closed class excludes K_t, its Assouad–Nagata dimension is at most $4^t - 1$.

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2023])

Every proper minor-closed class has asymptotic dimension at most 2.

My contribution:

Theorem (Distel [2023], Liu [2023])

Every proper minor-closed class has Assouad–Nagata dimension at most 2.

Some intuition

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

- Start with a graph embedded on a surface

Some intuition

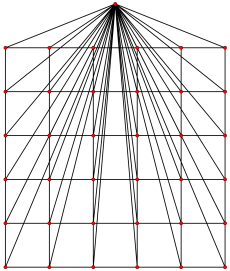
The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

- Start with a graph embedded on a surface
- Add a bounded number of “vortices”

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

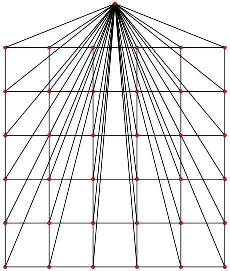
- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices



Some intuition

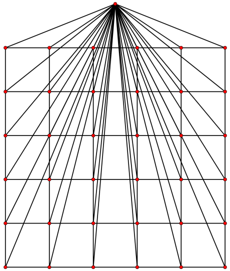
The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices
- Paste such graphs together via small cliques



Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).



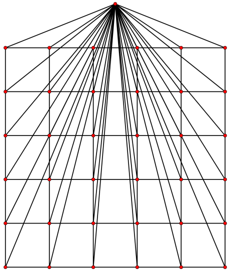
- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices
- Paste such graphs together via small cliques

We end up with something that is "not far" from an embeddable graph.

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices
- Paste such graphs together via small cliques

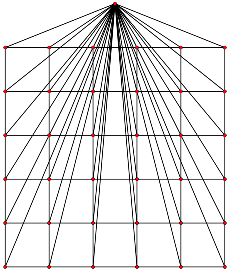


We end up with something that is "not far" from an embeddable graph.

We find that asymptotic and Assouad–Nagata "ignore" these modifications.

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).



- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices
- Paste such graphs together via small cliques

We end up with something that is "not far" from an embeddable graph.

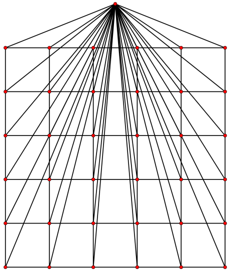
We find that asymptotic and Assouad–Nagata "ignore" these modifications.

This is somewhat unexpected

Some intuition

The structure of proper minor-closed classes is known (Robertson and Seymour [2003]).

- Start with a graph embedded on a surface
- Add a bounded number of "vortices"
- Add a bounded number of apex vertices
- Paste such graphs together via small cliques



We end up with something that is "not far" from an embeddable graph.

We find that asymptotic and Assouad–Nagata "ignore" these modifications.

This is somewhat unexpected; adding a single vertex to a planar graph can force the genus by an arbitrary amount.

What about non-minor-closed classes

What about non-minor-closed classes

We can't say much in the most general case

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

To *subdivide* an edge e in a graph G is to create a new graph in which the edge e has been replaced by a path.

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

To *subdivide* an edge e in a graph G is to create a new graph in which the edge e has been replaced by a path.

Theorem

Any class that is not contained within a proper minor-closed class and is closed under subdivision has infinite Assouad–Nagata dimension.

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

To *subdivide* an edge e in a graph G is to create a new graph in which the edge e has been replaced by a path.

Theorem

Any class that is not contained within a proper minor-closed class and is closed under subdivision has infinite Assouad–Nagata dimension.

The reason is that we have "too much" flexibility

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

To *subdivide* an edge e in a graph G is to create a new graph in which the edge e has been replaced by a path.

Theorem

Any class that is not contained within a proper minor-closed class and is closed under subdivision has infinite Assouad–Nagata dimension.

The reason is that we have "too much" flexibility, any control function would allow us to find a control function for every class of graphs.

What about non-minor-closed classes

We can't say much in the most general case, but there is a specific case we can answer.

To *subdivide* an edge e in a graph G is to create a new graph in which the edge e has been replaced by a path.

Theorem

Any class that is not contained within a proper minor-closed class and is closed under subdivision has infinite Assouad–Nagata dimension.

The reason is that we have "too much" flexibility, any control function would allow us to find a control function for every class of graphs.

This result gives an alternative proof for 1-planar graphs having infinite Assouad–Nagata dimension.

That’s it!

Thanks for listening!

Patrice Assouad. Sur la distance de Nagata. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(1):31–34, 1982.

G. Bell and A. Dranishnikov. Asymptotic dimension. *Topology Appl.*, 155(12): 1265–1296, 2008.

Marthe Bonamy, Nicolas Bousquet, Louis Esperet, Carla Groenland, Chun-Hung Liu, François Pirot, and Alex Scott. Asymptotic dimension of minor-closed families and Assouad-Nagata dimension of surfaces. *J. Euro. Math. Society*, 2023.

Marc Distel. Proper minor-closed classes of graphs have Assouad-Nagata dimension 2. 2023.

Anna Erschler and Ivan Mitrofanov. Assouad-Nagata dimension and gap for ordered metric spaces, 2021. arXiv:2109.12181.

M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

Chun-Hung Liu. Assouad-Nagata dimension of minor-closed metrics, 2023.

J. Nagata. Note on dimension theory for metric spaces. *Fund. Math.*, 45:143–181, 1958.

Mikhail I. Ostrovskii and David Rosenthal. Metric dimensions of minor excluded graphs and minor exclusion in groups. *Internat. J. Algebra Comput.*, 25(4): 541–554, 2015.

N. Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *J. Combin. Theory Ser. B*, 89(1):43–76, 2003.