Erdős-Ko-Rado theorems for finite general linear groups

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Joint work with Kai-Uwe Schmidt

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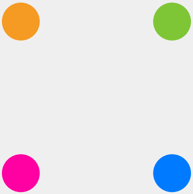
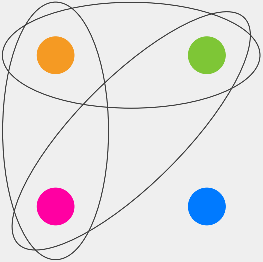


INTERSECTING k-SETS OF $[n]$



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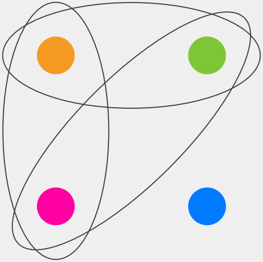

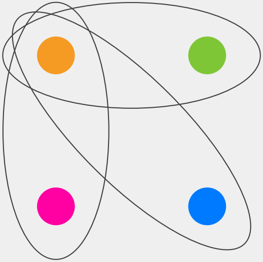

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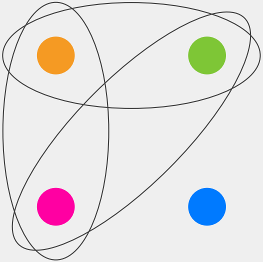

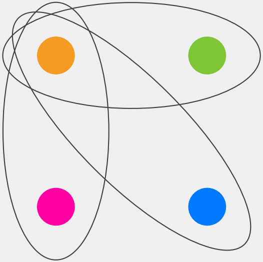

Theorem (Wilson 1984)

For n sufficiently large compared to k and t, a t-intersecting family of k-subsets of $[n]$ has size at most $\binom{n-t}{k-t}$.
If equality holds, then all members of the family contain a fixed t-subset of $[n]$.

Intersecting sets in $\mathcal{S}_n$

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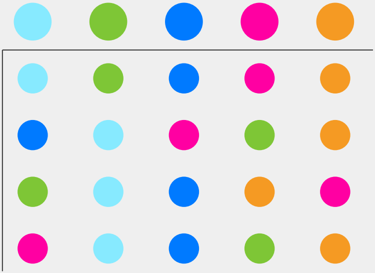

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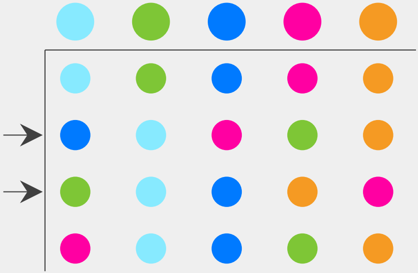

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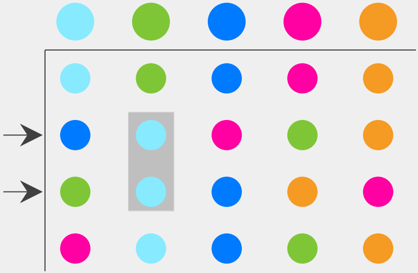

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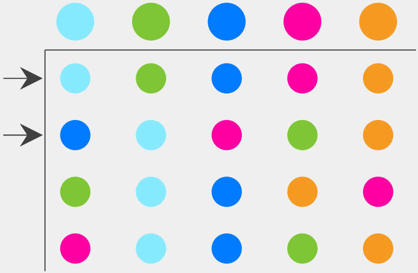

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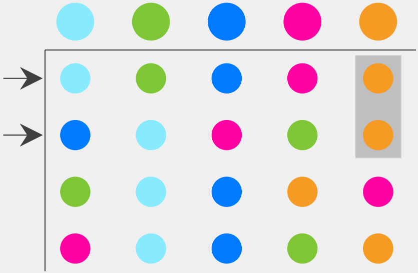
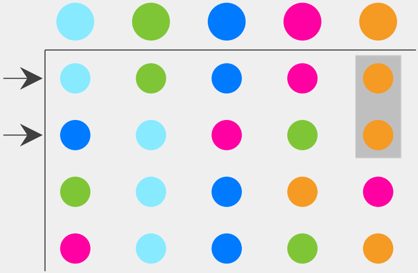
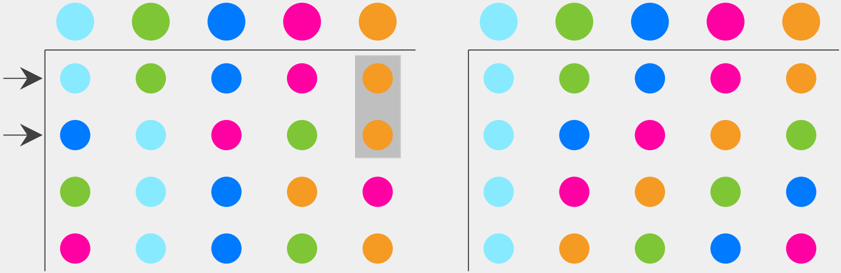




intersecting set in $\mathcal{S}_5$

INTERSECTING SETS IN $\mathcal{S}_n$



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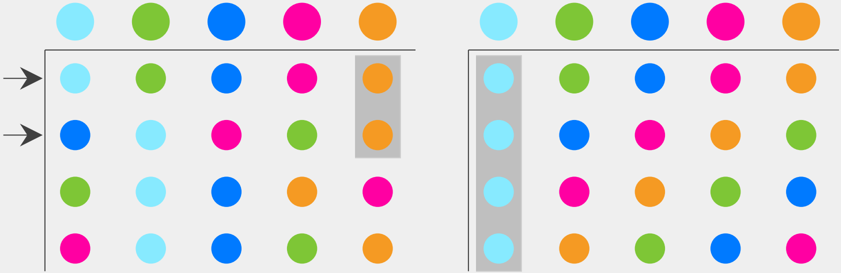

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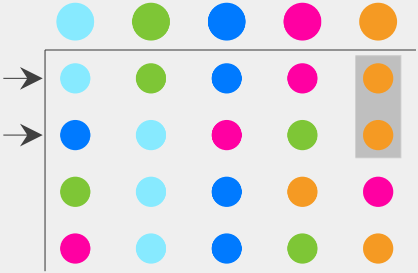
Intersecting sets in $\mathcal{S}_n$



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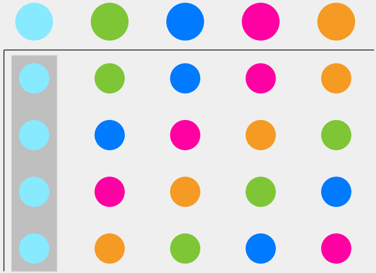

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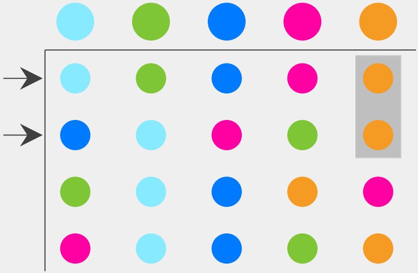

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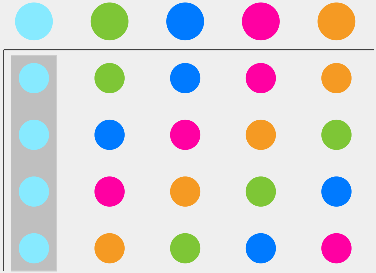

intersecting set in $\mathcal{S}_5$

Example

A coset of the stabiliser of an element in $[n]$ is intersecting and has size $(n-1)!$.

Intersecting sets in $\mathcal{S}_n$

Theorem (Deza, Frankl 1977)

The size of an intersecting set in $\mathcal{S}_n$ is at most $(n-1)!$.

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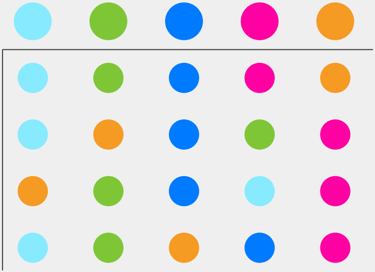
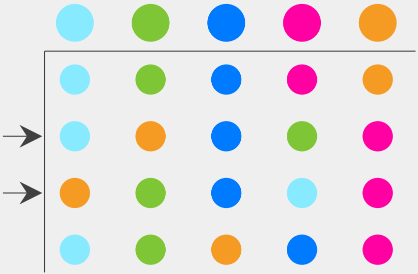
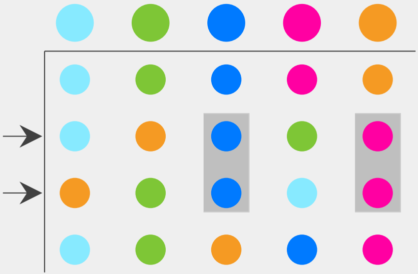
Theorem (Cameron, Ku 2003; Larose, Malvenuto 2004)

If an intersecting set in $\mathcal{S}_n$ is of maximal size, then it is a coset of the stabiliser of a point in $[n]$.

t-INTERSECTING SETS IN $\mathcal{S}_n$







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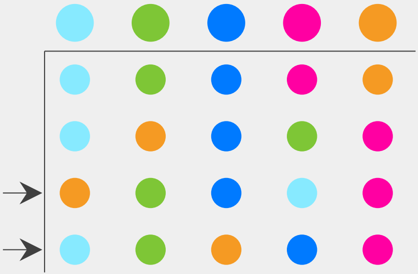
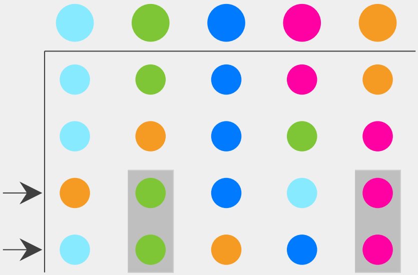
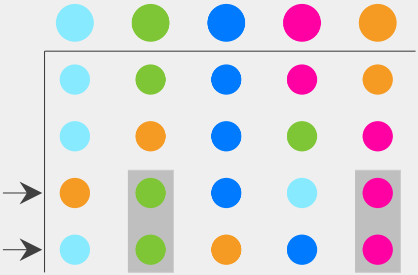
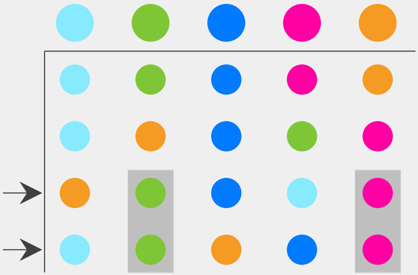






2-intersecting set in $\mathcal{S}_5$.



2-intersecting set in $\mathcal{S}_5$.

Example

A coset of the stabiliser of t distinct elements of $[n]$ is t-intersecting of size $(n-t)!$.

Conjecture (Deza, Frankl 1977)

If n is sufficiently large compared to t, then a t-intersecting set Y in $\mathcal{S}_n$ has size at most $(n-t)!$.
If equality holds, then Y is a coset of the stabiliser of t distinct elements of $[n]$.

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Theorem (Ellis, Friedgut, Pilpel 2011)

The conjecture is true.

t-INTERSECTING SETS IN $\mathrm{GL}(n, q)$

$$\mathbb{F}_2^3 = \langle \bullet, \bullet, \bullet \rangle = \{\bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet\}$$


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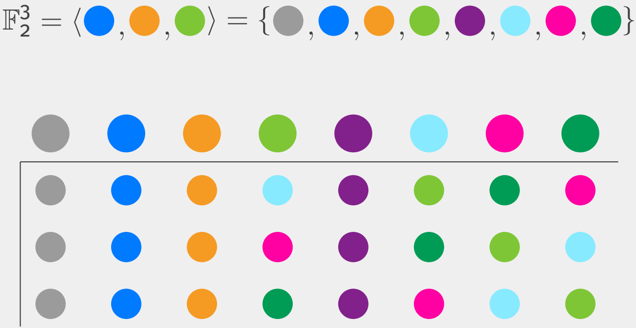
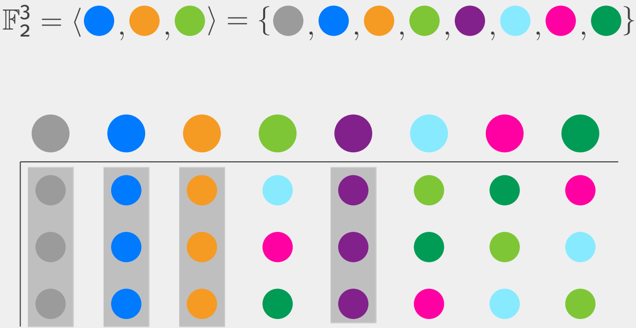
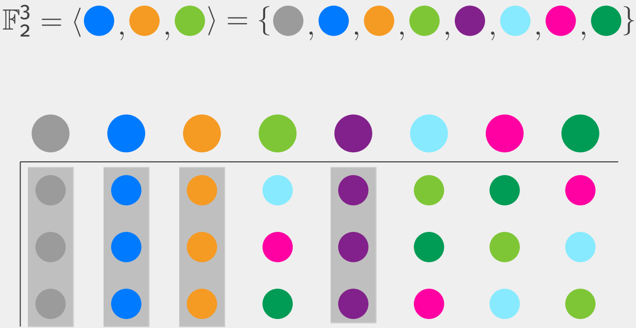
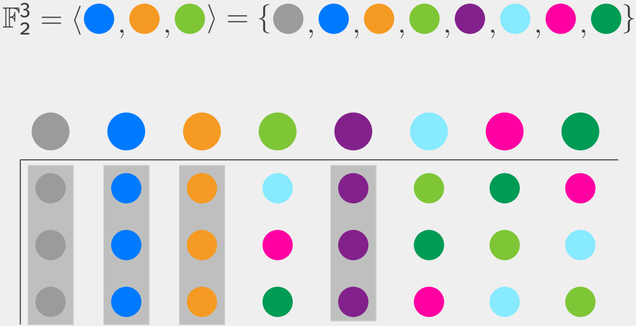






equal on q^2 elements

t-INTERSECTING SETS IN $\mathrm{GL}(n, q)$



equal on q^2 elements
2-intersecting in $\mathrm{GL}(3, 2)$

A coset of the stabiliser of t linearly independent elements of $\mathbb{F}_q^n$ is called *t-coset*.

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Example

A t-coset is t-intersecting of size

$$\prod_{i=t}^{n-1}(q^n - q^i).$$

Theorem (M. Ahanjideh, N. Ahanjideh 2014)

The size of a 1-intersecting set in $\mathrm{GL}(n,q)$ is at most

$$\prod_{i=1}^{n-1}(q^n - q^i).$$

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Theorem (Maegher, Razafimahatratra 2021)

The characteristic vector of a 1-intersecting set of maximal size in $\mathrm{GL}(2, q)$ is spanned by the characteristic vectors of 1-cosets.

Theorem (E., Schmidt 2023)

Let Y be a t-intersecting set in $\mathrm{GL}(n, q)$. If n is sufficiently large compared to t, then

$$|Y| \leq \prod_{i=t}^{n-1} (q^n - q^i) \tag{❀}$$

and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of t-cosets.

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and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of t-cosets.

The bound (❀) was recently and independently obtained by Ellis, Kindler, and Lifshitz with completely different techniques.

Extremal t-intersecting sets in $\mathrm{GL}(n, q)$

Are the t-cosets the only t-intersecting sets in $\mathrm{GL}(n, q)$ of maximal size?

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Theorem (Ahanjideh 2022)

A 1-intersecting set of $\mathrm{GL}(2, q)$ of maximal size is a 1-coset or the transpose of a 1-coset.

Extremal t-intersecting sets in $\mathrm{GL}(n, q)$

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Conjecture

Let Y be t-intersecting in $\mathrm{GL}(n, q)$ of maximal size. If n is sufficiently large compared to t, then Y or Y^T is a t-coset.

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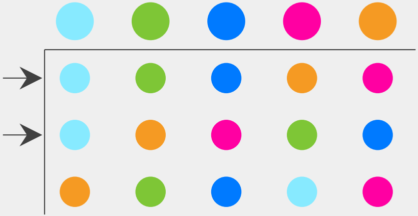
t-set-intersecting sets in $\mathcal{S}_n$

t-SET-INTERSECTING SETS IN $\mathcal{S}_n$



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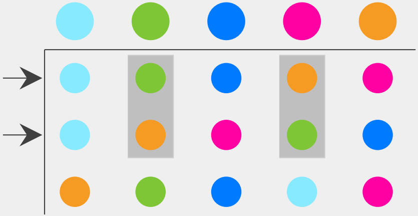

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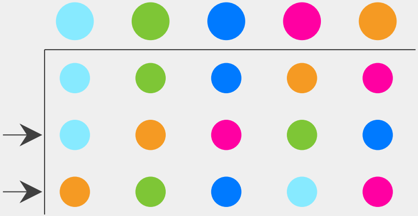

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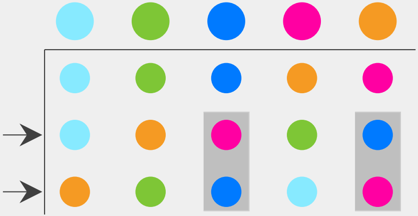
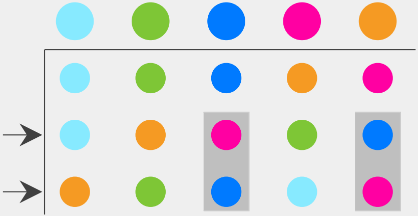
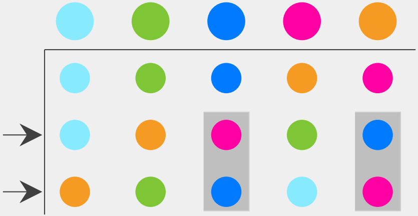




2-set-intersecting set in $\mathcal{S}_5$.



2-set-intersecting set in $\mathcal{S}_5$.

Example

A coset of the stabiliser of a t-set of $[n]$ is t-set-intersecting of size $t!(n-t)!$.

Theorem (Ellis 2012)

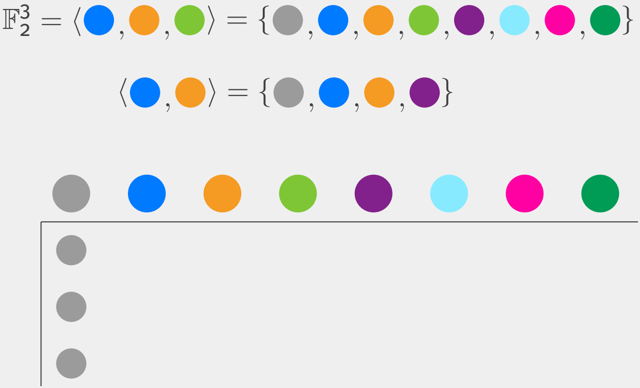
If n is sufficiently large compared to t, then a t-set-intersecting set Y in $\mathcal{S}_n$ has size at most $t!(n-t)!$.
If equality holds, then Y is a coset of the stabiliser of a t-set of $[n]$.

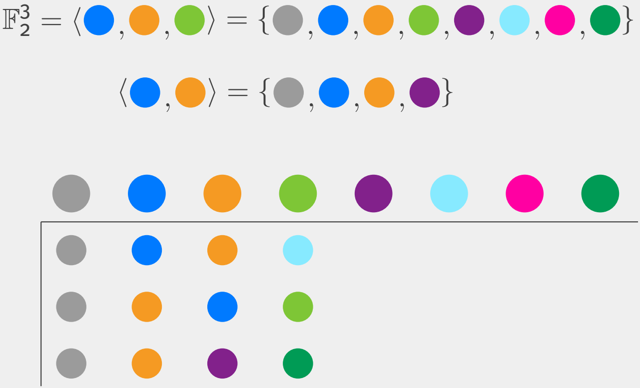
t-space-intersecting sets in $\mathrm{GL}(n,q)$

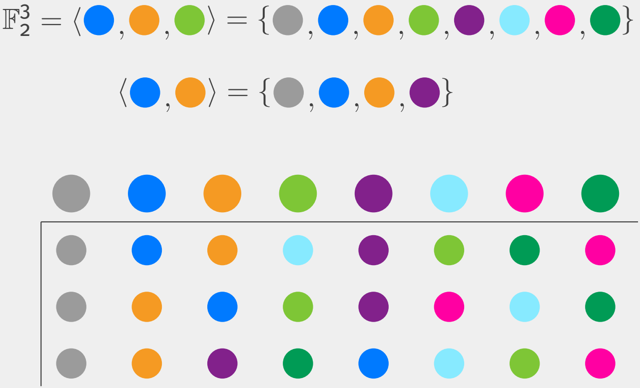
$$\mathbb{F}_2^3 = \langle {\color{blue}\bullet}, {\color{orange}\bullet}, {\color{green}\bullet} \rangle = \{ {\color{gray}\bullet}, {\color{blue}\bullet}, {\color{orange}\bullet}, {\color{green}\bullet}, {\color{purple}\bullet}, {\color{cyan}\bullet}, {\color{magenta}\bullet}, {\color{darkgreen}\bullet} \}$$

$$\langle {\color{blue}\bullet}, {\color{orange}\bullet} \rangle = \{ {\color{gray}\bullet}, {\color{blue}\bullet}, {\color{orange}\bullet}, {\color{purple}\bullet} \}$$

t-SPACE-INTERSECTING SETS IN $\mathrm{GL}(n,q)$







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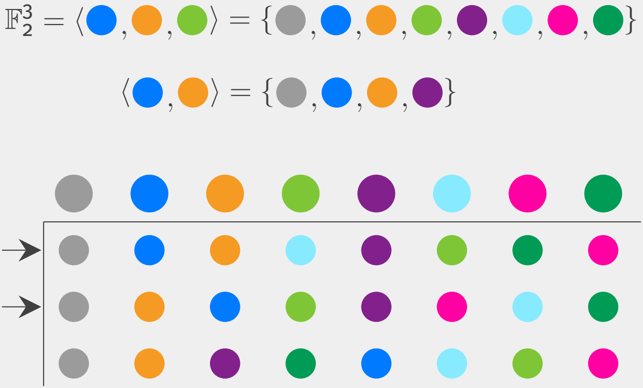

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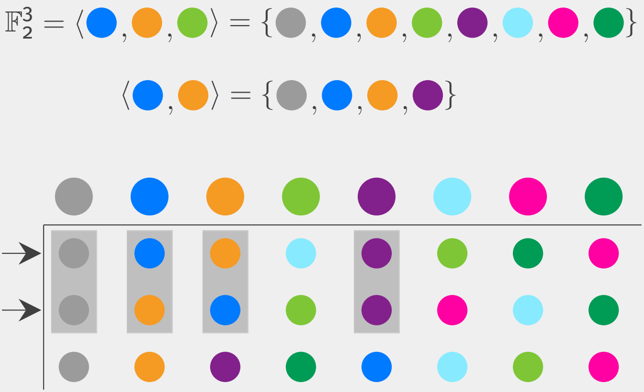
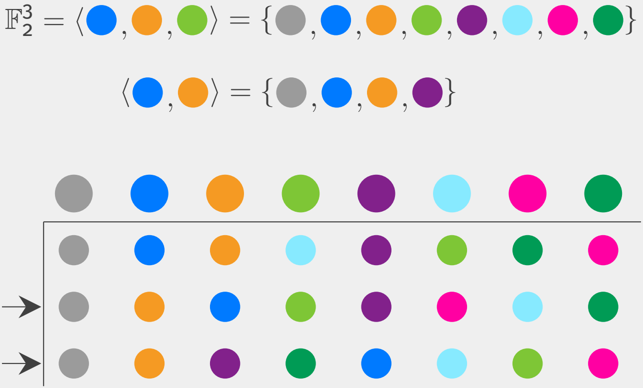
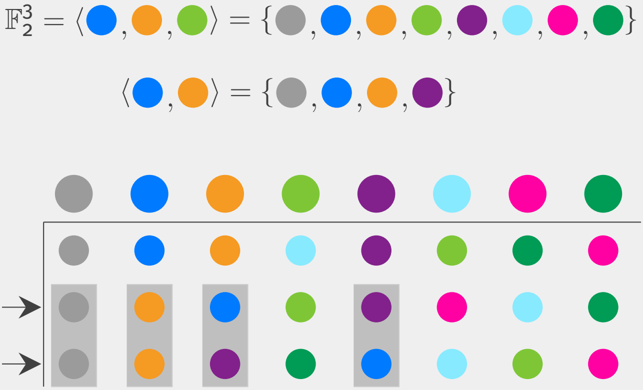
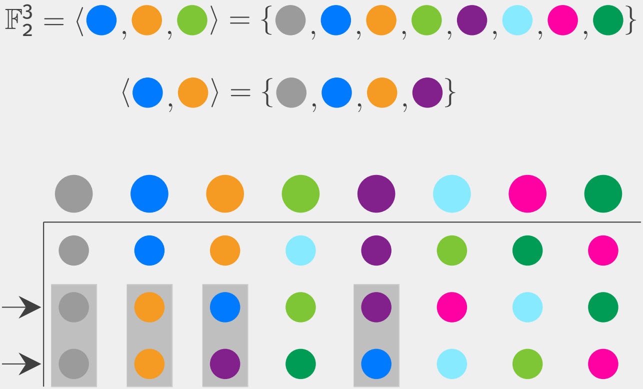






t-SPACE-INTERSECTING SETS IN $\mathrm{GL}(n, q)$



equal on a 2-space

$$\mathbb{F}_2^3 = \langle \textcolor{blue}{\bullet}, \textcolor{orange}{\bullet}, \textcolor{green}{\bullet} \rangle = \{ \textcolor{gray}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{orange}{\bullet}, \textcolor{green}{\bullet}, \textcolor{purple}{\bullet}, \textcolor{cyan}{\bullet}, \textcolor{magenta}{\bullet}, \textcolor{teal}{\bullet} \}$$

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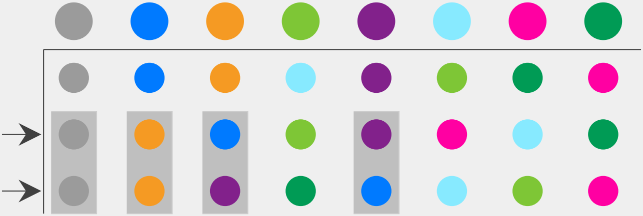
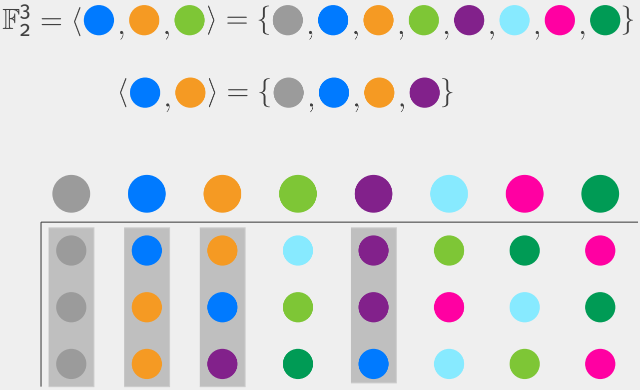


equal on a 2-space

2-space-intersecting in $\mathrm{GL}(3, 2)$



equal on a 2-space

2-space-intersecting in $\mathrm{GL}(3, 2)$

Example

A coset of the stabiliser of a t-space is t-space-intersecting of size

$$\left(\prod_{i=0}^{t-1}(q^t - q^i)\right)\left(\prod_{i=t}^{n-1}(q^n - q^i)\right).$$

Example

A coset of the stabiliser of a t-space is t-space-intersecting of size

$$\left(\prod_{i=0}^{t-1}(q^t - q^i)\right)\left(\prod_{i=t}^{n-1}(q^n - q^i)\right).$$

Theorem (Meagher, Spiga 2011)

A 1-space-intersecting set in $\mathrm{GL}(n,q)$ has size at most

$$(q-1)\prod_{i=1}^{n-1}(q^n - q^i).$$

Main theorem (2)

Theorem (E., Schmidt 2023)

Let Y be t-space-intersecting in $\mathrm{GL}(n,q)$. If n is sufficiently large compared to t, then

$$|Y| \leq \left(\prod_{i=0}^{t-1}(q^t - q^i)\right)\left(\prod_{i=t}^{n-1}(q^n - q^i)\right)$$

and, in case of equality, the characteristic vector of Y is spanned by the characteristic vectors of cosets of stabilisers of t-spaces.

Are the cosets of stabilisers of t-spaces the only t-space-intersecting sets in $\mathrm{GL}(n, q)$ of maximal size?

Extremal t-space-intersecting sets in $\mathrm{GL}(n, q)$

Are the cosets of stabilisers of t-spaces the only t-space-intersecting sets in $\mathrm{GL}(n, q)$ of maximal size?
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Conjecture

Let Y be t-space-intersecting in $\mathrm{GL}(n, q)$ of maximal size. If n is sufficiently large compared to t, then Y or Y^T is a coset of the stabiliser of a t-space.

Theorem (Ellis, Friedgut, Pilpel 2011)

Let $\Gamma = (X, E)$ be a graph and $\Gamma_0, \Gamma_1, \dots, \Gamma_r$ be regular spanning subgraphs of Γ with common eigenvectors $\{1, v_1, \dots, v_{n-1}\}$.

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Weighted version of Hoffman bound

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$$1_Y \in \langle \{1\} \cup \{v_k : P(k) = P_{\min}\} \rangle.$$

Conjugacy classes and irr. characters of $\mathrm{GL}(n, q)$ are indexed by

$$\underline{\sigma}\colon \{\text{ monic irr. polynomials}\} \setminus \{X\} \to \text{ Partitions}$$

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- Let $\Gamma_{\underline{\sigma}}$ be the graph with vertex set $\mathrm{GL}(n,q)$ and adjacency matrix

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Application of weighted Hoffman bound

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- Determine $\omega_{\underline{\sigma}}$ such that the sums $\sum_{\underline{\sigma}} \omega_{\underline{\sigma}} P_{\underline{\sigma}}(\underline{\lambda})$ have the required properties.

Thanks!