Spreading primitive groups of diagonal type do not exist

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Joint work with John Bamberg and Michael Giudici

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Bray, Cai, Cameron, Spiga & Zhang (2020): For affine and diagonal groups, synchronising $\iff$ separating.

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Theorem (Bamberg, Giudici, Lansdown & Royle, 2022)

The diagonal group $\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$ is synchronising for $q = 13$ and $q = 17$, and non-spreading for all q.

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Theorem (Bamberg, F. & Giudici, 2023+)

Each primitive group of diagonal type is non-spreading.

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Then $(*)$ holds $\iff$ $T \notin \{\mathrm{J}_1, \mathrm{M}_{22}, \mathrm{J}_3, \mathrm{McL}, \mathrm{Th}, \mathbb{M}\}$.

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$\implies$: GAP; Magma; Burness, O'Brien & Wilson (2010) for Th; Burness (2023) for $\mathbb{M}$.

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(P1) and (P2) $\implies$ $(*)$ $\implies$ $W(T)$ is non-spreading.

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We consider how the elements of N conjugate certain involutions of $A \setminus B$.

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For $\mathrm{O'N}$, we consider the right cosets of A in $\mathrm{Aut}(T)$, instead of in T.

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$\implies (X, J)$ is a witness to $W(T)$ being non-spreading.