# Levenshtein's Conjecture for Sequence Covering Arrays

Dani Gentle (she/they)

School of Mathematics,
Monash University

## Notation

For positive integers $v \geq t$:

- $[v] = \{0, \ldots, v-1\}$
- $S_v$ denotes the set of permutations of $[v]$
- $S_{v,t}$ denotes the set of ordered $t$-sequences of distinct elements of $[v]$
- E.g. $012 \in S_{5,3}$, $011 \notin S_{5,3}$, $0123 \notin S_{5,3}$

# Sequence Covering Arrays

A sequence $s \in S_{v,t}$ is covered by a permutation $\pi \in S_v$ if the elements of $s$ appear in $\pi$ in the order specified by $s$

012 and 014 are both covered by 01234

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**Definition**
A set of permutations $X \subseteq S_v$ with $N = |X|$ is a *sequence covering array*, denoted $\mathrm{SCA}(N; v, t)$, if for every sequence $s \in S_{v,t}$, there exists $\pi \in X$ that covers $s$.

- $v$ is the *order* of $X$.
- $t$ is the *strength* of $X$.

# Big question

- For given $v$ and $t$, what is the smallest $N$ such that a $\mathrm{SCA}(N; v, t)$ exists?
- For what values for $v$ and $t$ does there exist a $\mathrm{SCA}(t!; v, t)$?

**Theorem (Levenshtein, 1992)**
For $t \geq 3$, a $\mathrm{SCA}(t!; t+1, t)$ exists.

**Conjecture (Levenshtein, 1992)**
For $t \geq 3$, if a $\mathrm{SCA}(t!; v, t)$ exists, then $v \in \{t, t+1\}$.

# Levenshtein's conjecture

- A $SCA(24; 6, 4)$ does exist (Mathon and van Trung, 1999). This the only known counter-example to Levenshtein's conjecture.
- A $SCA(24; 7, 4)$ does not exist (Klein, 2004).
- The conjecture has been verified for $t \in \{3, 5, 6\}$.

**Theorem (Chee et al., 2013)**
For $t \geq 3$, if a $SCA(t!; v, t)$ exists, then $v \leq 2t - 1$.

# Interactions

Let A be an $N \times k$ array with entries from $[v]$. A 2-*way interaction* is a set of 2 pairs

$$T = \{(c_1, \nu_1), (c_2, \nu_2)\},$$

where $c_1$ and $c_2$ are distinct columns of A and $\nu_1$ and $\nu_2$ are elements of $[v]$.

A covers $T$ if there exists a row $r$ of A such that $\mathrm{A}[r, c_1] = \nu_1$ and $\mathrm{A}[r, c_2] = \nu_2$.

# Interaction covering arrays

**Definition**
An *interaction covering array*, denoted $\mathrm{ICA}(N; 2, k, v)$, is an $N \times k$ array with entries from $[v]$ that covers all 2-way interactions.

| $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |

**Table 1:** A strength 2 interaction covering array

# Excess coverage

Let $T = \{(c_1, \nu_1), (c_2, \nu_2)\}$. Then $T$ is either

- a constant pair, if $\nu_1 = \nu_2$, or
- a non-constant pair, if $\nu_1 \neq \nu_2$.

An interaction covering array has *excess coverage*, denoted $\mathsf{ICA}_X(N; 2, k, v)$, if each constant pair is covered at least twice.

**Theorem (Chee et al., 2013)**
For $v \geq 4$, If an $\mathsf{ICA}_X(v(v+1); 2, k, v)$ exists, then $k \leq v + 2$.

# Connection to sequence covering arrays

Let $S$ be a $\text{SCA}(t!; v, t)$.

Choose a sequence in $x \in S_{v,t-2}$ and let $R \subset S$ be the set of $t(t-1)$ permutations that cover $x$.

Construct a $t(t-1) \times (v-t+2)$ array, A, whose rows and columns are indexed by the elements of $R$ and the symbols not in $x$.

$\text{A}[r, c]$ is the number of symbols in $x$ that appear to the left of $c$ in the permutation $r$. Then A is an $\text{ICA}_X(t(t-1); 2, v-t+2, t-1)$.

# Orthogonal arrays

A row is *flat* if it contains only one symbol.

An *orthogonal array* is an ICA in which every interaction is covered exactly once.

An $\mathrm{ICA}_X(v(v+1); 2, k, v)$ contains an orthogonal array if and only if it has flat rows for each symbol.

**Corollary (G.)**
An $\mathrm{ICA}_X(v(v+1); 2, v+1, v)$ exists when $v$ is a prime power.

# OAs and non-OAs

**Lemma (G.)**
If $v!/(v-t+2)! > t!$, and a $\mathrm{SCA}(t!, v, t)$ exists, then a non-OA $\mathrm{ICA}_X$ must exist.

If $v = t + 2$, this inequality holds for $t \geq 4$.

# Existence of OAs and non-OAs

| $v$ | Maximum number of columns | |
|---|---|---|
| | OA | non-OA |
| 3 | 4 | 4 |
| 4 | 5 | 5 |
| 5 | 6 | 5 |
| 6 | 3 | 5 |

These results were found by computer search.

The non-existence of an $\mathrm{ICA}_X(42; 2, 6, 6)$ shows a $\mathrm{SCA}(7!; 11, 7)$ does not exist.

# $\mathbf{ICA}_X(42; 2, 5, 6)$

There is a unique $\mathrm{ICA}_X(42; 2, 5, 6)$.

- It's a non-OA,
- No flat rows,
- Every row contains a repeated symbol.

# Non-existence of a SCA$(7!; 10, 7)$

The existence of an $\mathrm{ICA}_X(42; 2, 5, 6)$ suggests a SCA$(7!; 10, 7)$ could exist.

**Theorem (G.)**
A SCA$(7!; 10, 7)$ does not exist.

**Proof.**
Suppose a SCA$(7!; 10, 7)$ exists and contains the identity.

Consider the $\mathrm{ICA}_X(42; 2, 5, 6)$ built from the sequence $x = 01234$.

This array should have a flat row but no such $\mathrm{ICA}_X$ exists.

$\square$

# Non-existence of a SCA$(7!; 10, 7)$

Alternatively,

**Theorem (G.)**
A SCA$(7!; 10, 7)$ does not exist.

**Proof.**
Suppose a SCA$(7!; 10, 7)$ exists and contains the identity.

Consider the ICA$_X(42; 2, 5, 6)$ built from the sequence $x = 02468$.

This array should have a row with no repeated symbols but no such ICA$_X$ exists.

$\square$

## Future research

- Is the max number of columns for an $\mathrm{ICA}_X$ $v+1$ or $v+2$?
- Find a general construction for non-OA $\mathrm{ICA}_X$.
- Explore connections between SCAs and $\mathrm{ICA}_X$ to make further progress on Levenshtein's conjecture.

Thank you!