Latin Squares with Restricted Transversals

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Latin squares

Definition (Latin square)

A *latin square* is an $n \times n$ array consisting of n distinct symbols where each symbol appears exactly once in each row and each column. One can use $\mathbb{Z}_n$ for indexing rows, columns and symbols of a latin square of order n.

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Example

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

Transversals in latin squares

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These two transversals are **not** disjoint:

0	**1**	2	3
1	0	**3**	2
2	3	0	1
3	2	1	**0**

0	**1**	2	3
1	0	3	**2**
2	3	**0**	1
3	2	1	0

Orthogonal mates

Definition (Orthogonal mates)

A pair of latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ of order n are said to be *orthogonal mates* if the n^2 ordered pairs (a_{ij}, b_{ij}) are distinct.

Example

$B =$

2	3	1
1	2	3
3	1	2

$A =$

2	1	3
1	3	2
3	2	1

(2,2)	(3,1)	(1,3)
(1,1)	(2,3)	(3,2)
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Remark

For orthogonal mates A and B, it is simple to see that if we look at all n occurrences of a given symbol in B, then the corresponding positions in A must form a transversal.

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Euler observed that the addition table of $\mathbb{Z}_n$, where n is even, does not contain any transversals.

0	1	2	3
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Latin squares without disjoint transversals

Theorem (Cavenagh and Wanless, 2017)

For even $n \to \infty$, there are at least $n^{n^{\frac{3}{2}}(\frac{1}{2}-o(1))}$ species of transversal-free latin squares of order n.

Main theorem

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Our main theorem is the following:

Theorem (Ghafari & Wanless, 2023+)

There exist arbitrarily large even order latin squares with at least one transversal, yet all transversals concide on $\lfloor \frac{n}{6} \rfloor$ entries, where n is the order of the latin square.

Orthogonal array representation

Remark

- Every latin square can be represented as a set of n^2 ordered triples (r, c, s), where the ordered pair (r, c) is the index of row and column of symbol s.
- The Latin property ensures that when two triples are not identical, they must share at most one coordinate.

Example

0	1	2
1	2	0
2	0	1

$\{(0,0,0),(0,1,1),(0,2,2),(1,0,1),(1,1,2),(1,2,0),(2,0,2),$
$(2,1,0),(2,2,1)\}.$

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$(2,1,0),(2,2,1)\}.$

Δ-lemma

The following lemma will be crucial to prove our main theorem.

Lemma (Δ-lemma)

Let L be a latin square of order n indexed by $\mathbb{Z}_n$. Define a function $\Delta : L \longrightarrow \mathbb{Z}_n$ by $\Delta(r, c, s) = s - r - c$. If T is a transversal of L then, modulo n,

$$\sum_{(r,c,s)\in T} \Delta(r, c, s) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{2}n, & \text{if } n \text{ is even}. \end{cases}$$

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Proof.
We split it into three cases. We denote the family for the case of $n = 6k$, where $k \geq 2$, by $\mathcal{P}_n$.

Proof of Theorem

For $n = 6k$, where $k \geq 3$ is an integer, consider latin square $\mathcal{P}_n$ given by:

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$$\mathcal{P}_n[a,b] = \begin{cases} a-2 & \text{if } (a,b) = (3,0) \\ a+2 & \text{if } (a,b) \in \{(1,0),(2,1)\} \\ a+2b & \text{if } (a,b) \in \{(0,1),(0,2)\} \\ a+b+3 & \text{if } b \equiv 1 \mod 3 \text{ and } a = 0 \\ a+b-3 & \text{if } \ b > 1,\ b \equiv 1 \mod 3 \ \text{ and } \ a = 3 \\ a+b-2 & \text{if } a > 3,\ \ a \equiv 0 \mod 3 \text{ and } b \equiv 0 \mod 2 \\ a+b+2 & \text{if } a > 3,\ \ a \equiv 1 \mod 3 \text{ and } b \equiv 0 \mod 2,\ b \notin \{n-2a+3, n-2a+2\} \\ a+b+1 & \text{if } (a > 3,\ \ a \equiv 1 \mod 3 \text{ and } b \in \{n-2a+3, n-2a+2\}) \ \textit{or} \\ & (a > 3,\ \ a \equiv 2 \mod 3 \text{ and } b = n-2a+4) \\ a+b-1 & \text{if } (a > 3,\ \ a \equiv 2 \mod 3 \text{ and } b \in \{n-2a+5\}) \ \textit{or} \\ & ((a,b) \in \{(1,1),(1,2),(2,2),(3,1)\}) \\ a+b & \text{otherwise.} \end{cases}$$

Proof of Theorem

Recall the Δ-lemma for even n:

$$\sum_{(r,c,s)\in T} \Delta(r,c,s) = \frac{1}{2}n \mod n.$$

The corresponding non-zero Δ-values of $\mathcal{P}_n$, where $n = 6 \times 4$, is as follows:

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	0	1	2	4	7	10	12	13	16	18	19	20	22
0		1	2	3	3	3		3	3		3		3
1	2	-1	-1										
2		1	-1										
3	-2	-1		-3	-3	-3		-3	-3		-3		-3
4	2		2	2		2			2	1	1	2	2
5										1	-1		
6	-2		-2	-2		-2	-2			-2		-2	-2
7	2		2	2		2	1	1		2		2	2
8							1	-1					
9	-2		-2	-2		-2	-2		-2	-2		-2	-2

Proof of Theorem

This leads us to the following lemma:

Lemma

The latin square $\mathcal{P}_n$ has a transversal, and all of them include the entries $(1,0),(2,1),(5,n-6),(8,n-12),\ldots,(3k-4,12)$.

Proof of Theorem

It can be verified that the following corresponds to a transversal

$$col(a) = \begin{cases} 4 & \text{if } a = 0 \\ a - 1 & \text{if } a \in \{1, 2, 3\} \\ 5 & \text{if } a = 3k + 2 \\ n - 2a + 4 & \text{if } (4 \leq a \leq 3k - 3 \text{ and } a \not\equiv 0 \bmod 3) \text{ or } a = 3k - 2 \text{ or } a = 3k - 1 \\ n - 2a + 7 & \text{if } 4 \leq a \leq 3k - 3 \text{ and } a \equiv 0 \bmod 3 \\ n - 2a + 3 & \text{if } (a \geq 3k + 3 \text{ and } a \equiv 0 \bmod 3) \text{ or } a = 3k \\ n - 2a + 6 & \text{if } a \geq 3k + 3 \text{ and } a \equiv 1 \bmod 3 \\ n - 2a + 9 & \text{if } (a \geq 3k + 3 \text{ and } a \equiv 2 \bmod 3) \text{ or } a = 3k + 1 \end{cases}$$

Hence, the proof is complete.

Proof of Theorem

We denote the family for the case of $n = 6k + 4$ by $\mathcal{L}_n$.

Recall the Δ-lemma for even n:

$$\sum_{(r,c,s)\in T} \Delta(r, c, s) = \frac{1}{2}n \mod n.$$

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	0	1	2	5	8	10	11	12	14	16	17	18	20
0		1	3	3	3		3		3		3		3
1	2	-1	-1										
3	-2		-2	-3	-3		-3		-3		-3		-3
4	2		2		2	2		2	2	1	1	2	2
5										1	-1		
6	-2		-2		-2	-2		-2	-2	-2		-2	-2
7	2		2		2	1	1	2	2	2		2	2
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9	-2		-2		-2	-2		-2	-2	-2		-2	-2

Latin squares of odd orders

- We didn't find any analogue of the mentioned results for odd orders. We don't know if there is any latin square with any number of disjoint transversals less than $\frac{n}{3} + 2$.

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- Wanless and Zhang, 2013 showed there are latin squares of order $n = 3k$, where $k \geq 4$ is an integer, with no more than $\frac{n}{3} + 2$ disjoint transversals.

Latin squares of odd orders

Theorem (Ghafari & Wanless, 2022)

For all odd $m \geq 3$, there exists a latin square of order $n = 3m$ with three subsquares of size m where every transversal has to hit each of these subsquares at least once.

References

Cavenagh, N. J., & Wanless, I. M. (2017). Latin squares with no transversals. *The Electronic Journal of Combinatorics*, *24*(2), P2–45.

Wanless, I. M., & Zhang, X. (2013). Transversals of latin squares and covering radius of sets of permutations. *European Journal of Combinatorics*, *34*(7), 1130–1143.

Thank you!