How to design a graph with three eigenvalues

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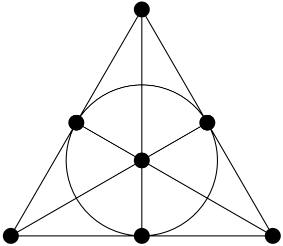

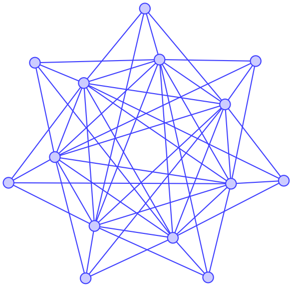

Plan

- Graphs with three distinct eigenvalues
- Coherent closure of a graph
- Graphs with small coherent rank
- Graphs with two valencies and large coherent rank
- Graphs with three valencies

Graphs with few distinct eigenvalues

- 1 distinct eigenvalue:

Graphs with few distinct eigenvalues

- 1 distinct eigenvalue: no edges (empty graphs)

$$(0)$$

Graphs with few distinct eigenvalues

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$$\begin{pmatrix} 0 \end{pmatrix}$$

- 2 distinct eigenvalues:

Graphs with few distinct eigenvalues

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- 2 distinct eigenvalues: all edges (complete graphs)

$$\begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} = J - I$$

Graphs with few distinct eigenvalues

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Graphs with few distinct eigenvalues

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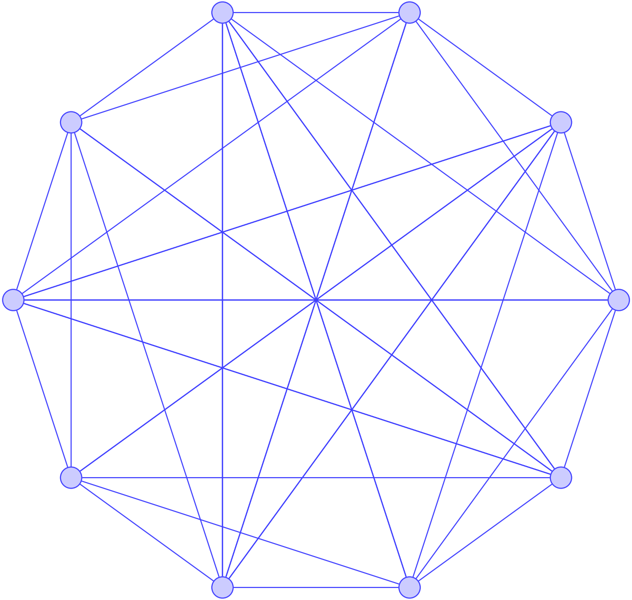
- 3 distinct eigenvalues: ...

Regular graphs with three eigenvalues



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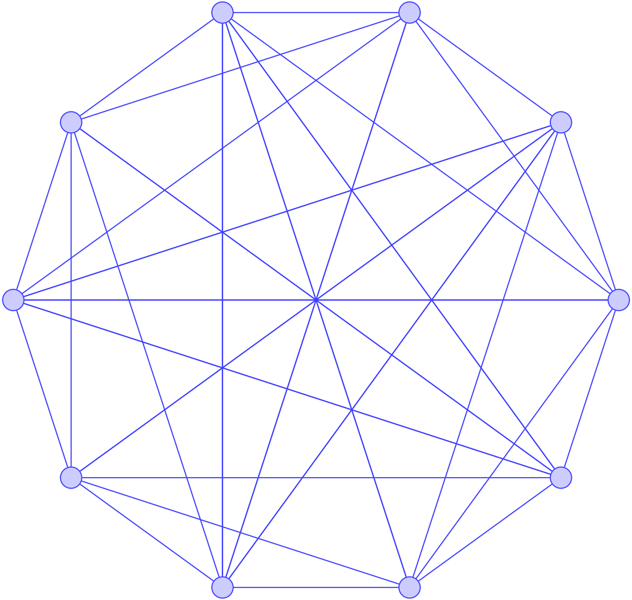

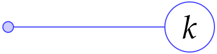

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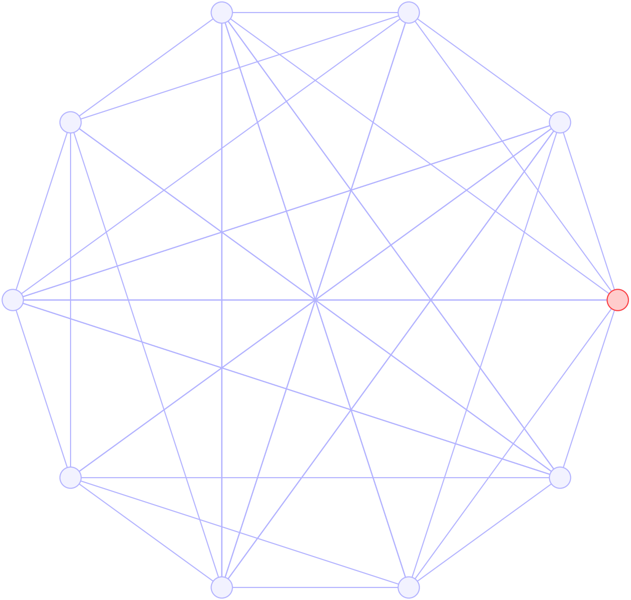

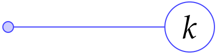

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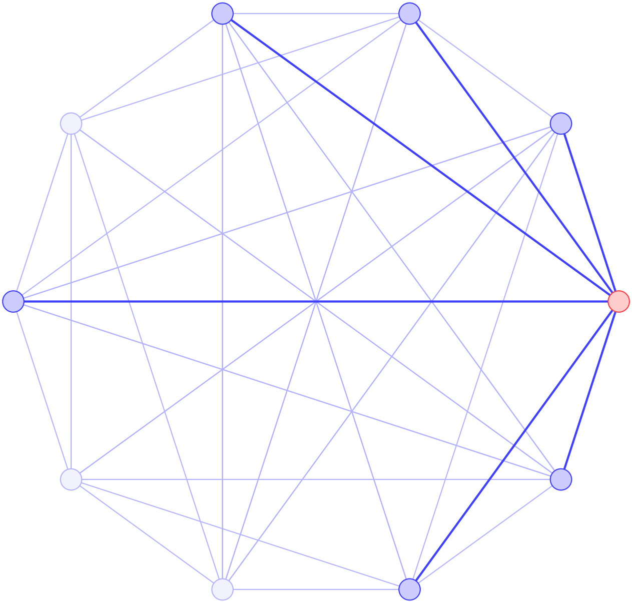

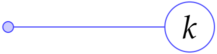

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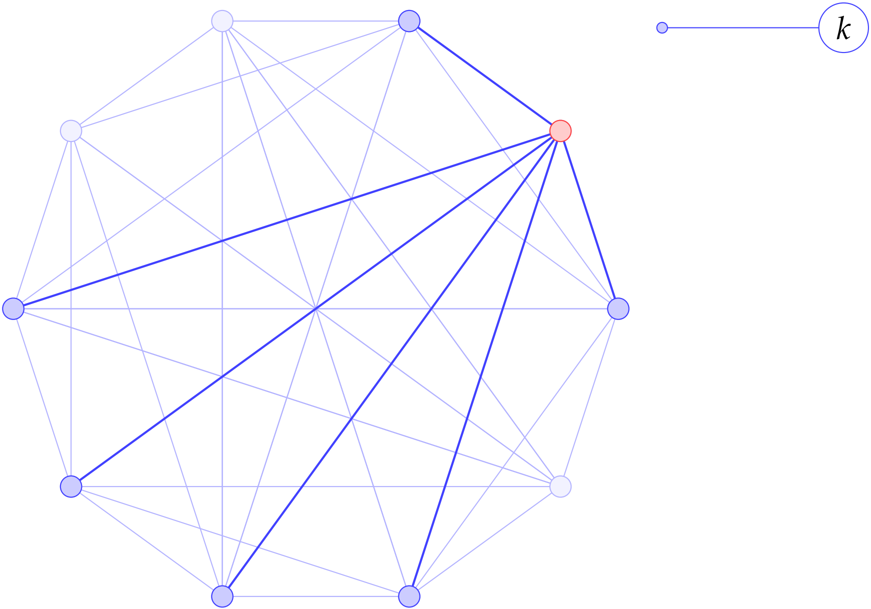

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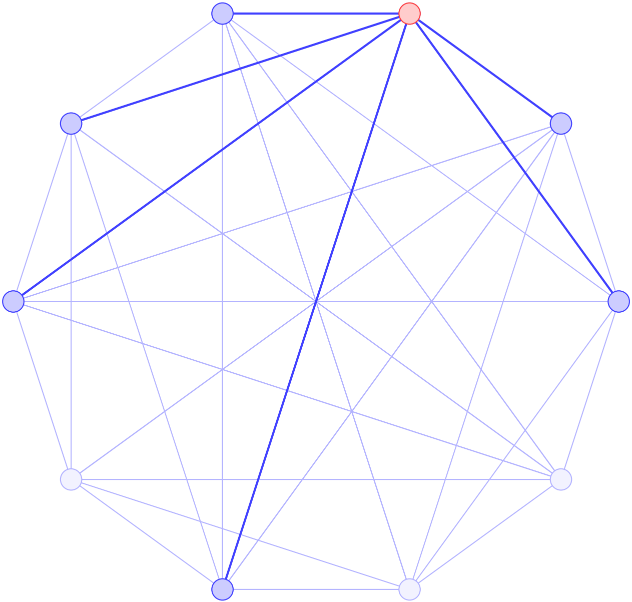

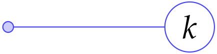

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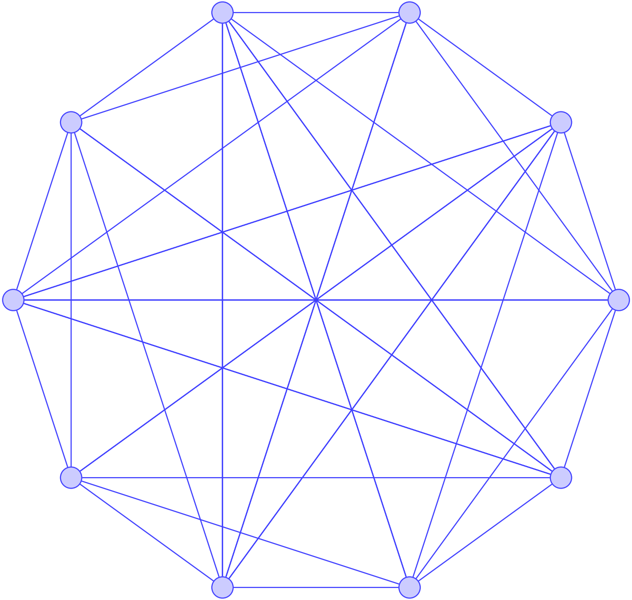
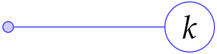
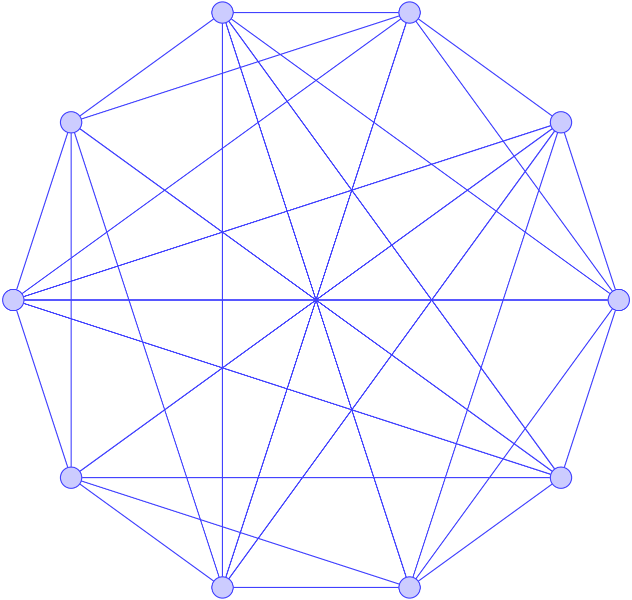
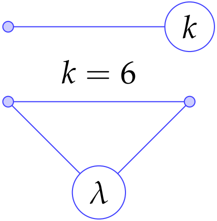




$k = 6$

Regular graphs with three eigenvalues





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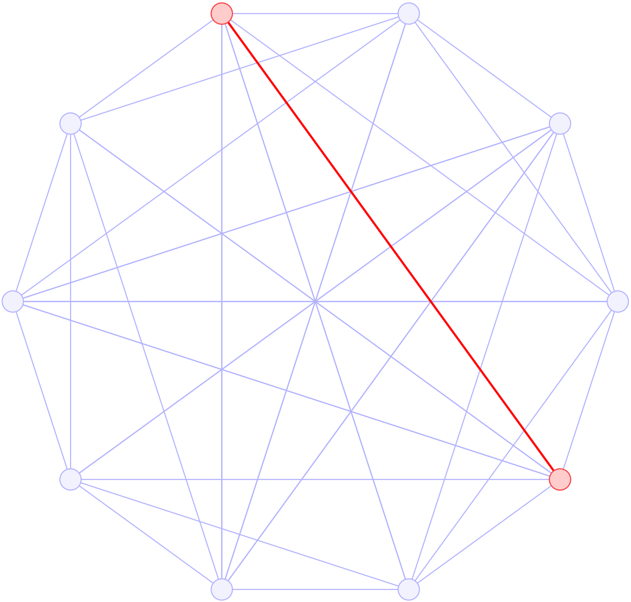

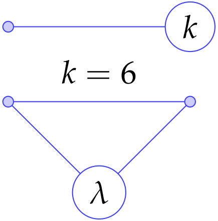

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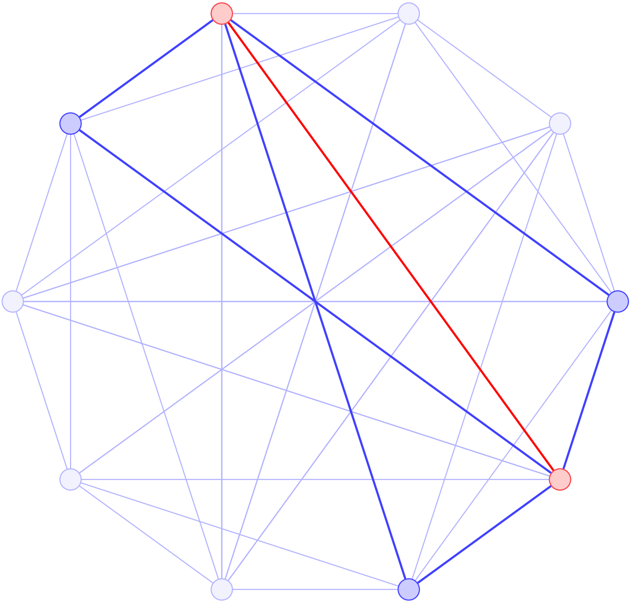

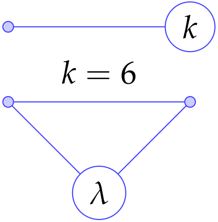

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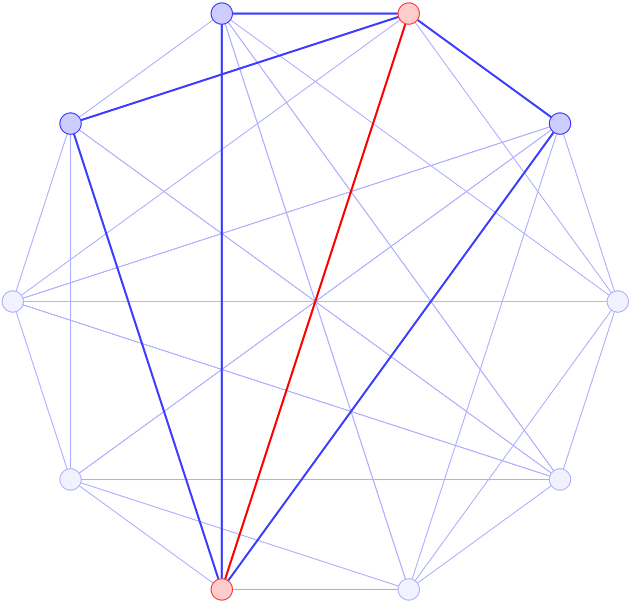

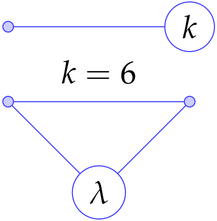

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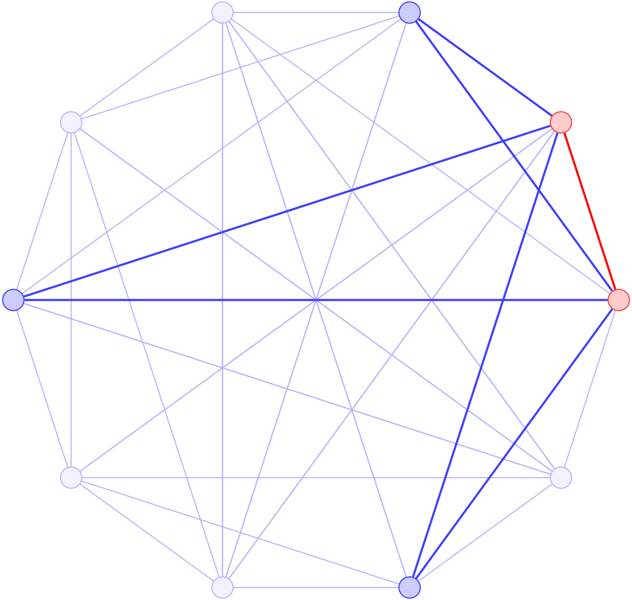

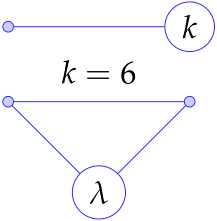

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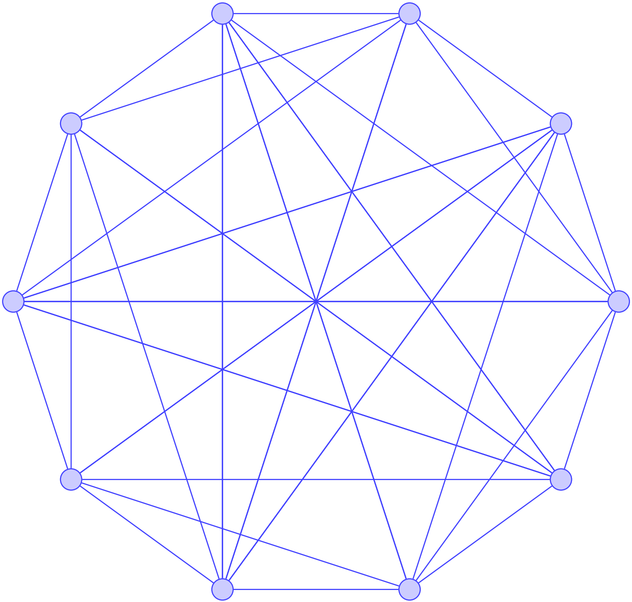

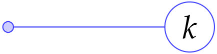

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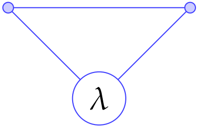
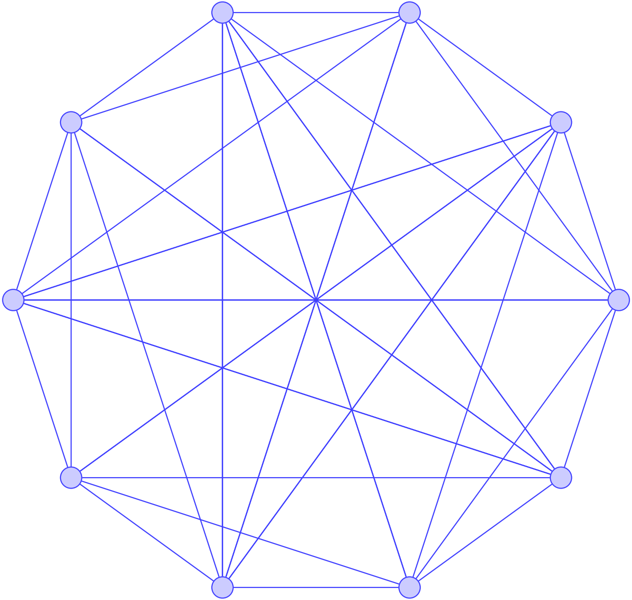
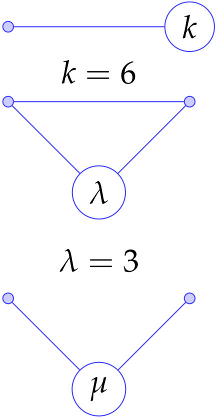


$\lambda = 3$

Regular graphs with three eigenvalues





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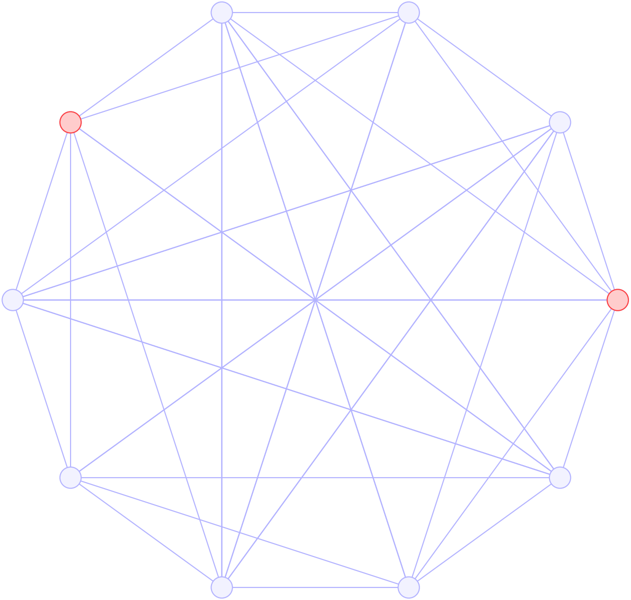

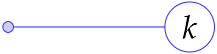

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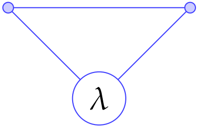

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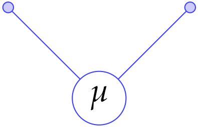

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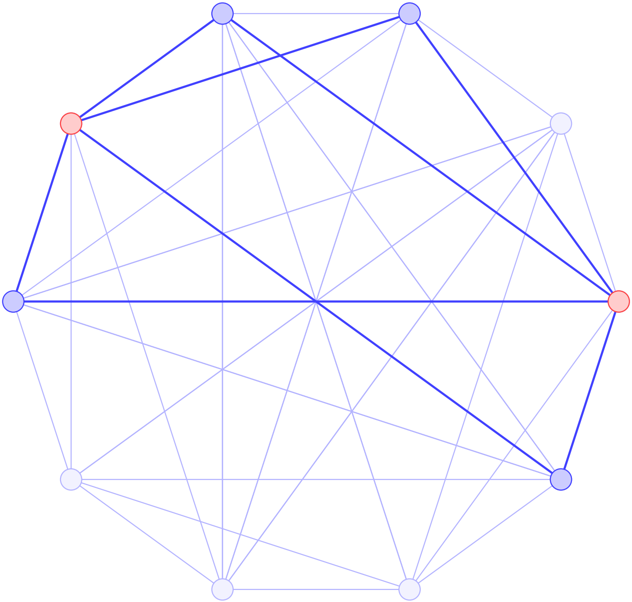

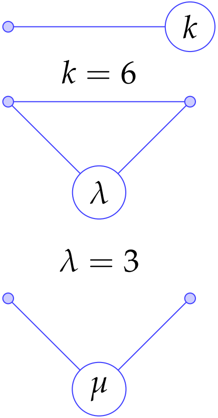

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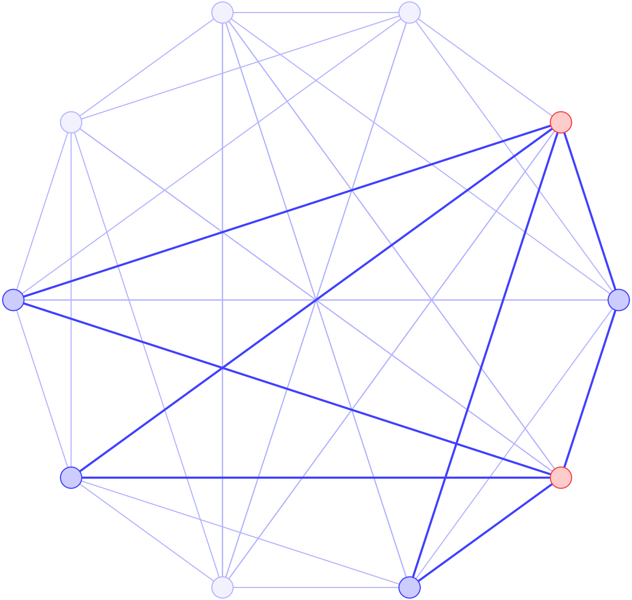

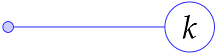

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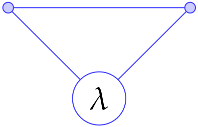

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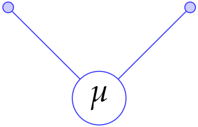

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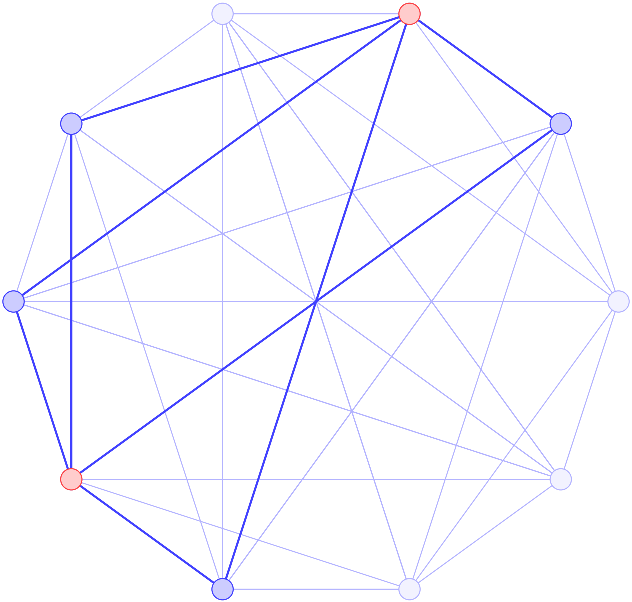

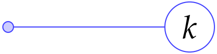

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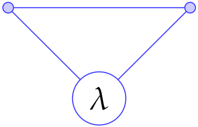

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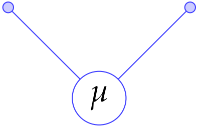
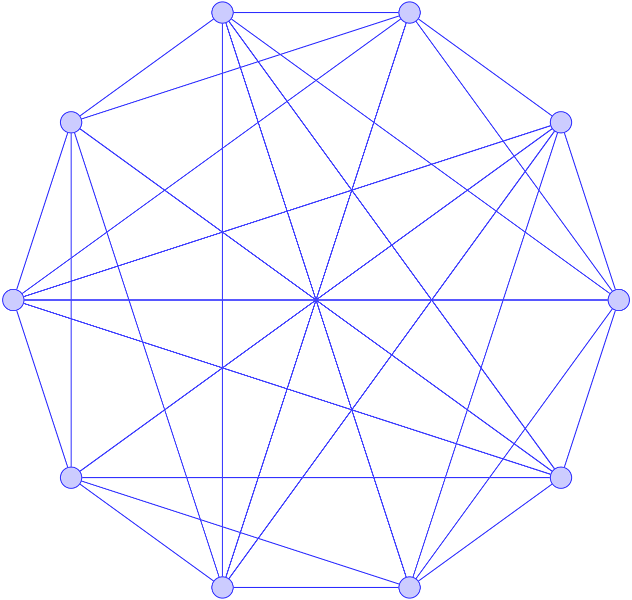
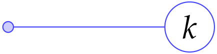


Regular graphs with three eigenvalues





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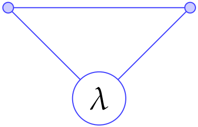

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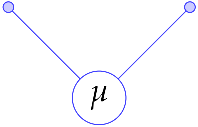
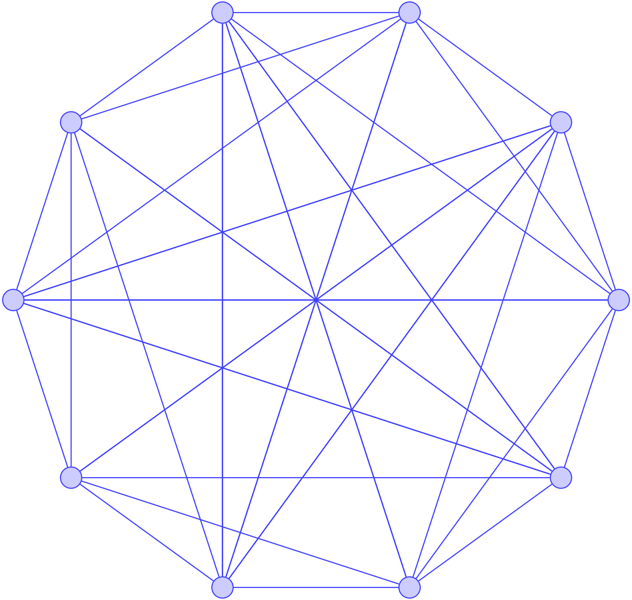
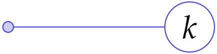


$\mu = 4$

Regular graphs with three eigenvalues





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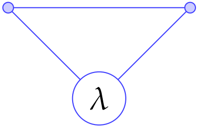

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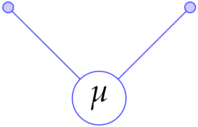


$\mu = 4$

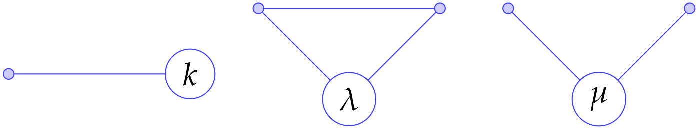
strongly regular

$\mathrm{srg}(10, 6, 3, 4)$

Regular graphs with three eigenvalues

A: adjacency matrix of $\mathrm{srg}(v, k, \lambda, \mu)$.

$$A^2 = kI + \lambda A + \mu(J - I - A)$$



$$\Updownarrow$$

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

Nonregular graphs with three eigenvalues

Question (Haemers 1995)
Apart from strongly regular graphs and complete bipartite graphs, *which graphs have just three distinct eigenvalues?*

Nonregular graphs with three eigenvalues

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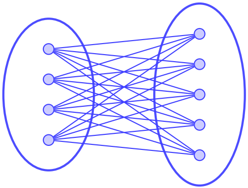
Strongly regular graph: a regular graph (V, E) for which $\exists \lambda, \mu$ such that, $\forall x, y \in V$ with $x \neq y$, the number of common neighbours of x and y is

$$\begin{cases} \lambda, & \text{if } x \sim y \\ \mu, & \text{if } x \not\sim y. \end{cases}$$

Complete bipartite graphs $K_{a,b}$ have spectrum

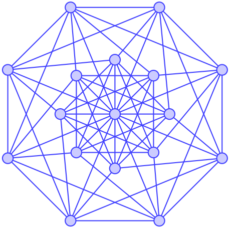
$$\left\{ [\sqrt{ab}]^1, [0]^{a+b-2}, [-\sqrt{ab}]^1 \right\}.$$



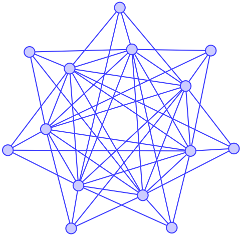
Nonregular graphs with three eigenvalues

Question (Haemers 1995)
Apart from strongly regular graphs and complete bipartite graphs, *which graphs have just three distinct eigenvalues?*

Muzychuk-Klin (1998): infinite families of examples with
Van Dam (1998): **two valencies** and a positive number of examples with **three valencies**.



Shrikhande cone



Fano graph

Cones over strongly regular graphs

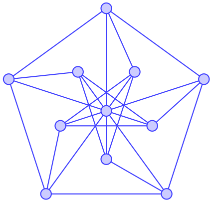
Cone: n-vertex graph with a vertex of valency $n-1$.
Cone over Γ: join of K_1 and Γ.

Theorem (Muzychuk and Klin 1998).
Let Γ be a (non-complete) strongly regular graph with v vertices, valency k, and smallest eigenvalue $-m$. The cone over Γ has precisely three distinct eigenvalues if and only if

$$m(k+m) = v.$$

Petersen cone:

$$\{[5]^1, [1]^5, [-2]^5\}$$



Switching strongly regular graphs

Switching: $\begin{bmatrix} A & B \\ B^\intercal & C \end{bmatrix} \mapsto \begin{bmatrix} A & J-B \\ J^\intercal - B^\intercal & C \end{bmatrix}$

	$\mathrm{srg}(v,k,\lambda,\mu)$	switched spectrum
Muzychuk-Klin	$(36,14,7,4)$	$\{[21]^1,[5]^7,[-2]^{28}\}$
Martin	$(105,72,51,45)$	$\{[60]^1,[9]^{21},[-3]^{83}\}$
Van Dam	$(176,49,12,14)$	$\{[61]^1,[5]^{97},[-7]^{78}\}$
Van Dam	$(256,105,44,42)$	$\{[121]^1,[9]^{104},[-7]^{151}\}$
Van Dam	$(126,45,12,18)$	$\{[57]^1,[3]^{89},[-9]^{36}\}$
$\vdots$		$\vdots$

Suppose A, C have orders v_A, v_C and $A\mathbf{1} = a\mathbf{1}$, $C\mathbf{1} = c\mathbf{1}$.

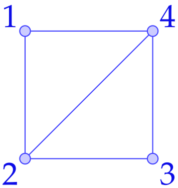
Works if $\begin{bmatrix} a & v_C-k+a \\ v_A-k+c & c \end{bmatrix}$ & $\mathrm{srg}(v,k,\lambda,\mu)$ share an eigenvalue.

Van Dam, JCTB (1998)
Muzychuk and Klin, Discrete Math (1998)

Coherent closure

[Weisfeiler-Leman stabilisation]

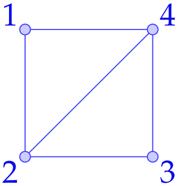
$$M_1 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{b} & \boxed{a} & \boxed{b} & \boxed{b} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{b} & \boxed{b} & \boxed{b} & \boxed{a} \end{bmatrix}$$



Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$

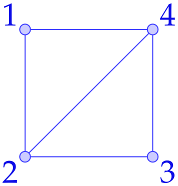


$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{b} & \boxed{a} & \boxed{b} & \boxed{b} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{b} & \boxed{b} & \boxed{b} & \boxed{a} \end{bmatrix}$$



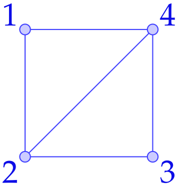
$$M_1^2 = \begin{bmatrix} a^2 + 2b^2 + c^2 & ab + ba + b^2 + cb & ac + 2b^2 + ca & ab + ba + b^2 + cb \\ ab + ba + b^2 + bc & a^2 + 3b^2 & ab + ba + b^2 + bc & ab + ba + 2b^2 \\ ac + 2b^2 + ca & ab + ba + b^2 + cb & a^2 + 2b^2 + c^2 & ab + ba + b^2 + cb \\ ab + ba + b^2 + bc & ab + ba + 2b^2 & ab + ba + b^2 + bc & a^2 + 3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \boxed{a} & & & \\ & & & \\ & & \boxed{a} & \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{b} & \boxed{a} & \boxed{b} & \boxed{b} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{b} & \boxed{b} & \boxed{b} & \boxed{a} \end{bmatrix}$$



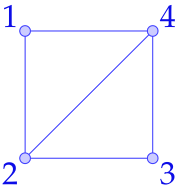
$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & & \boxed{b} \\ & & & \\ & \boxed{b} & \boxed{a} & \boxed{b} \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{b} & \boxed{a} & \boxed{b} & \boxed{b} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{b} & \boxed{b} & \boxed{b} & \boxed{a} \end{bmatrix}$$



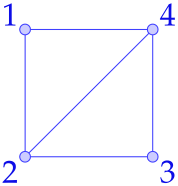
$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ & & & \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$



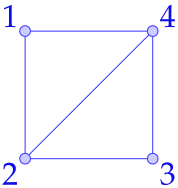
$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ \color{red}{ab+ba+b^2+bc} & a^2+3b^2 & \color{red}{ab+ba+b^2+bc} & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ \color{red}{ab+ba+b^2+bc} & ab+ba+2b^2 & \color{red}{ab+ba+b^2+bc} & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} a & b & c & b \\ d & & d & \\ c & b & a & b \\ d & & d & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$



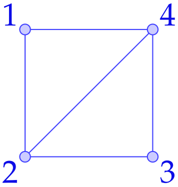
$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} a & b & c & b \\ d & e & d & \\ c & b & a & b \\ d & & d & e \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_1 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{b} & \boxed{a} & \boxed{b} & \boxed{b} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{b} & \boxed{b} & \boxed{b} & \boxed{a} \end{bmatrix}$$



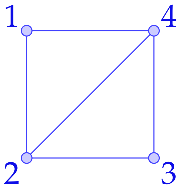
$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

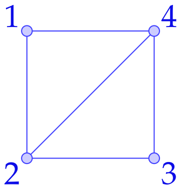
$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$

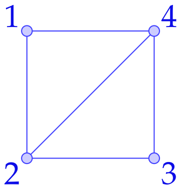


$$M_2^2 = \begin{bmatrix} a^2 + 2bd + c^2 & ab + be + bf + cb & ac + 2bd + ca & ab + be + bf + cb \\ da + dc + ed + fd & 2db + e^2 + f^2 & da + dc + ed + fd & 2db + ef + fe \\ ac + 2bd + ca & ab + be + bf + cb & a^2 + 2bd + c^2 & ab + be + bf + cb \\ da + dc + ed + fd & 2db + ef + fe & da + dc + ed + fd & 2db + e^2 + f^2 \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



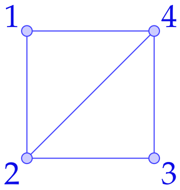
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & & & \\ & & & \\ & & \boxed{a} & \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



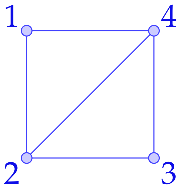
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & \boxed{b} & & \boxed{b} \\ & & & \\ & \boxed{b} & \boxed{a} & \boxed{b} \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



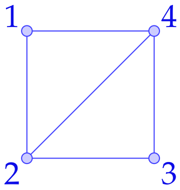
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ & & & \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ & & & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



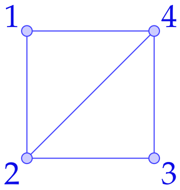
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ \color{red}{da+dc+ed+fd} & 2db+e^2+f^2 & \color{red}{da+dc+ed+fd} & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ \color{red}{da+dc+ed+fd} & 2db+ef+fe & \color{red}{da+dc+ed+fd} & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & & \boxed{d} & \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & & \boxed{d} & \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



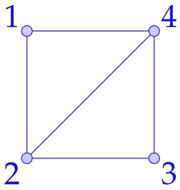
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} a & b & c & b \\ d & e & d & \\ c & b & a & b \\ d & & d & e \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



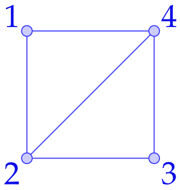
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$

Coherent closure

[Weisfeiler-Leman stabilisation]

$$M_2 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$

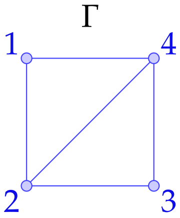


$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix} = M_2$$

Coherent closure

$$\mathcal{W}(\Gamma) = \begin{bmatrix} \boxed{a} & \boxed{b} & \boxed{c} & \boxed{b} \\ \boxed{d} & \boxed{e} & \boxed{d} & \boxed{f} \\ \boxed{c} & \boxed{b} & \boxed{a} & \boxed{b} \\ \boxed{d} & \boxed{f} & \boxed{d} & \boxed{e} \end{bmatrix}$$



$$\overset{\boxed{a}}{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}, \overset{\boxed{b}}{\begin{bmatrix} 0&1&0&1\\0&0&0&0\\0&1&0&1\\0&0&0&0 \end{bmatrix}}, \overset{\boxed{c}}{\begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}}, \overset{\boxed{d}}{\begin{bmatrix} 0&0&0&0\\1&0&1&0\\0&0&0&0\\1&0&1&0 \end{bmatrix}}, \overset{\boxed{e}}{\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}}, \overset{\boxed{f}}{\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}}$$

We say "Γ has **coherent rank** 6".

Coherent rank of graphs with three eigenvalues

Theorem (Muzychuk-Klin 1998)
Let Γ be a connected graph w/ three distinct eigenvalues. Then the coherent rank of Γ is

- $= 3$ iff Γ is strongly regular;
- $\neq 4$;
- $= 5$ iff $\Gamma \cong K_{1,b}$ with $b > 1$;
- $= 6$ iff $\Gamma \cong K_{a,b}$ with $2 \leqslant a < b$ or
  Γ is a cone over a strongly regular graph;
- $\neq 7$.

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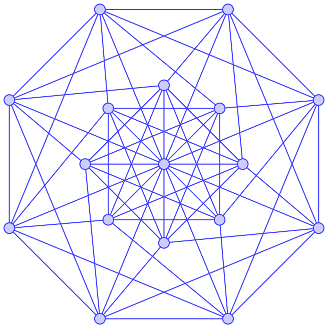
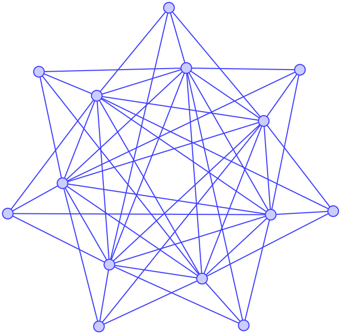
Proposition 6.2. *For a non-standard graph Γ the cases* $\dim(W(\Gamma)) = r$, $r \in \{7, 8\}$ *are impossible.*

non-standard: connected w/ 3 evs, not srg, not $K_{a,b}$.



Shrikhande cone



Fano graph

Coherent rank: 6

Coherent rank: ??

✓

X

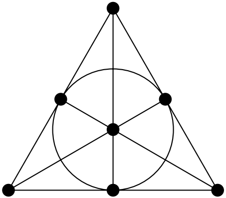
Symmetric designs

Definition (2-design)

- **points:** $X = \{1, \ldots, v\}$;
- **blocks:** $\mathcal{B} \subset \binom{X}{k}$;
- every pair $\{x, y\} \in \binom{X}{2}$ is contained in λ blocks.

Then $(X, \mathcal{B})$ is called a 2-(v, k, λ) **design**.

If $|\mathcal{B}| = v$ then $(X, \mathcal{B})$ is called **symmetric**.



Fano plane:

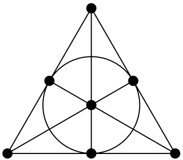
symmetric 2-$(7, 3, 1)$ design

Total graph of a symmetric design

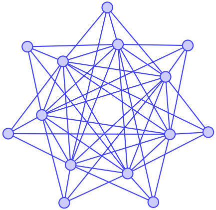
B: incidence matrix of a symmetric 2-design $\mathcal{D}$.

Total graph of $\mathcal{D}$: $\begin{bmatrix} O & B \\ B^\intercal & J-I \end{bmatrix}$.

Theorem (Van Dam 1998)
Total graph of a symmetric 2-(q^3-q+1, q^2, q) design
spectrum: $\left\{ [q^3]^1, [q-1]^{(q-1)q(q+1)}, [-q]^{(q-1)q(q+1)+1} \right\}$.



Fano graph:
$(q=2)$



Graphs with coherent rank 8

B: incidence matrix of a symmetric 2-design $\mathcal{D}$.

Total graph of $\mathcal{D}$: $\begin{bmatrix} O & B \\ B^\intercal & J-I \end{bmatrix}$.

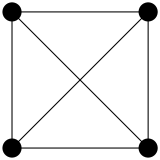
Theorem (GG and Yip 2023+)
Let Γ be a connected graph w/ three distinct eigenvalues. Then $\mathcal{W}(\Gamma)$ has rank 8 if and only if Γ is the total graph of a symmetric 2-(q^3-q+1, q^2, q) design.

$$\mathcal{W}(\Gamma) = \boxed{a}\begin{bmatrix} I & O \\ O & O \end{bmatrix} + \boxed{b}\begin{bmatrix} J-I & O \\ O & O \end{bmatrix} + \boxed{c}\begin{bmatrix} O & B \\ O & O \end{bmatrix} + \boxed{d}\begin{bmatrix} O & J-B \\ O & O \end{bmatrix}$$
$$+ \boxed{e}\begin{bmatrix} O & O \\ B^\intercal & O \end{bmatrix} + \boxed{f}\begin{bmatrix} O & O \\ J-B^\intercal & O \end{bmatrix} + \boxed{g}\begin{bmatrix} O & O \\ O & I \end{bmatrix} + \boxed{h}\begin{bmatrix} O & O \\ O & J-I \end{bmatrix}$$

Quasi-symmetric designs

Definition (quasi-symmetric design)
A 2-(v,k,λ) design $(X,\mathcal{B})$ is called **quasi-symmetric** if $\forall B_1 \neq B_2$ in $\mathcal{B}$ we have $|B_1 \cap B_2| \in \{x,y\}$ with $x \neq y$.

x and y are called **intersection numbers**.



quasi-symmetric 2-$(4,2,1)$ design

intersection numbers: 0 and 1

Definition (block graph)
The x-**block graph** of $(X,\mathcal{B})$ has vertex set $\mathcal{B}$, and two blocks are adjacent iff they intersect in x points.

Total graph of a quasi-symmetric design

B: incidence matrix of a quasi-symmetric 2-design $\mathcal{Q}$.
C: adjacency matrix of the x-block graph of $\mathcal{Q}$.

x-**total graph** of $\mathcal{Q}$: $\begin{bmatrix} O & B \\ B^\intercal & C \end{bmatrix}$.

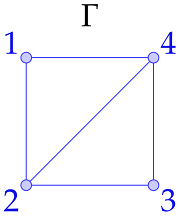
Theorem (Van Dam 1998)
The q-total graph of a quasi-symmetric 2-$(q^3, q^2, q+1)$ design with intersection numbers 0 and q has spectrum:

$$\left\{ [q^3+q^2+q]^1, [q]^{q^3-1}, [-q]^{q^3+q^2+q} \right\}.$$

- Case $q = 2$ discovered earlier by Bridges and Mena (1981)

Type of a coherent closure

$$\mathcal{W}(\Gamma) = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



$\downarrow$ reorder vertices $\downarrow$

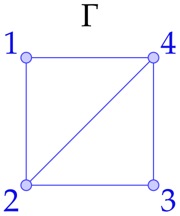
$$\begin{bmatrix} a & c & b & b \\ c & a & b & b \\ d & d & e & f \\ d & d & f & e \end{bmatrix}$$

$\mathcal{W}(\Gamma)$ has **type:**

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Type of a coherent closure

$$\mathcal{W}(\Gamma) = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



$\downarrow$ reorder vertices $\downarrow$

$$\begin{bmatrix} a & c & b & b \\ c & a & b & b \\ d & d & e & f \\ d & d & f & e \end{bmatrix}$$

$\mathcal{W}(\Gamma)$ has **type:**

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

coherent rank = sum of entries

Graphs with coherent rank 9

Theorem (GG and Yip 2023+)
Let Γ be a connected graph w/ three distinct eigenvalues. Then $\mathcal{W}(\Gamma)$ has rank 9 if and only if Γ or $\overline{\Gamma}$ is *a* total graph of certain quasi-symmetric 2-designs. (Type $\left[\begin{smallmatrix}2&2\\2&3\end{smallmatrix}\right]$)

- Muzychuk-Klin 1998: quasi-sym 2-$(8, 6, 15)$ design with intersection numbers 4 and 5.
- Van Dam 1998: quasi-sym 2-$(q^3, q^2, q+1)$ designs with intersection numbers 0 and q.

- GG and Yip 2023+: quasi-sym 2-$(22, 15, 80)$ design with intersection numbers 9 and 11.

Graphs with coherent rank 9

Theorem (GG and Yip 2023+)
Let Γ be a connected graph w/ three distinct eigenvalues. Then $\mathcal{W}(\Gamma)$ has rank 9 if and only if Γ or $\overline{\Gamma}$ is *a* total graph of certain quasi-symmetric 2-designs. (Type $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 3 \end{smallmatrix}\right]$)

$(v, k, \lambda; x, y)$	Spectrum	Exists
$(76, 40, 52; 24, 20)$	$\{[125]^1, [11]^{75}, [-5]^{190}\}$	?
$(120, 50, 35; 25, 20)$	$\{[153]^1, [9]^{119}, [-6]^{204}\}$	?
$(141, 45, 33; 9, 15)$	$\{[175]^1, [5]^{329}, [-13]^{140}\}$	?
$(121, 46, 69; 16, 21)$	$\{[368]^1, [5]^{483}, [-23]^{121}\}$	?
$(85, 40, 130; 15, 20)$	$\{[224]^1, [4]^{595}, [-31]^{84}\}$	?
$(225, 36, 10; 0, 6)$	$\{[384]^1, [9]^{224}, [-6]^{400}\}$	?
$(120, 75, 370; 50, 45)$	$\{[476]^1, [44]^{119}, [-6]^{952}\}$	?
$(232, 112, 296; 48, 56)$	$\{[539]^1, [7]^{1276}, [-41]^{231}\}$	?
$\vdots$	$\vdots$	$\vdots$

Switching strongly regular graphs

Switching: $\begin{bmatrix} A & B \\ B^\intercal & C \end{bmatrix} \mapsto \begin{bmatrix} A & J-B \\ J^\intercal - B^\intercal & C \end{bmatrix}$

	srg(v,k,λ,μ)	switched spectrum	rank
Muzychuk-Klin	$(36,14,7,4)$	$\{[21]^1,[5]^7,[-2]^{28}\}$	9
Van Dam	$(176,49,12,14)$	$\{[61]^1,[5]^{97},[-7]^{78}\}$	134
Van Dam	$(126,45,12,18)$	$\{[57]^1,[3]^{89},[-9]^{36}\}$	1222
Van Dam	$(256,105,44,42)$	$\{[121]^1,[9]^{104},[-7]^{151}\}$	2048
Martin	$(105,72,51,45)$	$\{[60]^1,[9]^{21},[-3]^{83}\}$	2893
Van Dam	$(625,288,133,132)$	$\{[313]^1,[13]^{287},[-12]^{337}\}$	15625
Van Dam	$(729,390,207,210)$	$\{[363]^1,[12]^{391},[-15]^{337}\}$	19683

Question: Is arbitrarily large rank possible?

Van Dam, JCTB (1998)
Muzychuk and Klin, Discrete Math (1998)

Switching Latin square graphs

Latin square: $n \times n$ matrix over $\{1, \dots, n\}$ s.t. each element occurs precisely once in each row and column.

X and Y are **orthogonal** if $|\{(X_{i,j}, Y_{i,j}) \mid 1 \leqslant i,j \leqslant n\}| = n^2$.

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&1&4&3\\\hline 3&4&1&2\\\hline 4&3&2&1\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 4&3&2&1\\\hline 2&1&4&3\\\hline 3&4&1&2\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&1&2\\\hline 4&3&2&1\\\hline 2&1&4&3\\\hline\end{array}.$$

Given mutually orthogonal Latin squares $L^{(1)}, L^{(2)}, \dots, L^{(m-2)}$, form graph $\mathcal{L}_m(n)$ with vertex set $\{(i,j) \mid 1 \leqslant i,j \leqslant n\}$ and

$(i,j) \sim (k,l)$ iff $\left(i, j, L^{(1)}_{i,j}, \dots, L^{(m-2)}_{i,j}\right)$ and $\left(k, l, L^{(1)}_{k,l}, \dots, L^{(m-2)}_{k,l}\right)$ agree in just 1 place.

$$\mathcal{L}_m(n) \in \mathrm{srg}(n^2, m(n-1), n-2+(m-1)(m-2), m(m-1)).$$

Switching Latin square graphs

Theorem (GG and Yip 2023+)
For $N = \frac{q^2}{2} - \frac{q\sqrt{3(q^2+2)}}{6}$, switching $\mathcal{L}_{\frac{q^2-1}{2}}(q^2)$ w.r.t. NK_{q^2} results in a graph w/ 3 distinct eigenvalues.

- q an odd prime power $\implies \mathcal{L}_{\frac{q^2-1}{2}}(q^2)$ exists.
- $q = a_k \implies N \in \mathbb{N}$, where: $a_k = 4a_{k-1} - a_{k-2}$ and $a_0 = 1,\ a_1 = 5$.

- Examples: $q = 5, 19, 71, 3691, 1911861, 138907099, \ldots$

- Hone et al. (2018) **conjecture** a_k is prime infinitely often.
- Shorey and Stewart (1983): a_k is a *proper* power for only finitely many k.

Graphs with three valencies

Theorem (GG and Yip 2023+)
Let Γ be connected w/ three distinct eigenvalues and three distinct valencies. Then $\mathrm{rank}(\mathcal{W}(\Gamma)) \geqslant 14$.

Valencies	Spectrum	Coherent rank
$\{[45]^1, [25]^{18}, [13]^{27}\}$	$\{[21]^1, [3]^{19}, [-3]^{26}\}$	16
$\{[15]^4, [10]^{16}, [7]^4\}$	$\{[11]^1, [3]^7, [-2]^{16}\}$	18
$\{[96]^1, [61]^{64}, [21]^{32}\}$	$\{[56]^1, [4]^{41}, [-4]^{55}\}$	20
$\{[24]^{18}, [14]^9, [8]^9\}$	$\{[20]^1, [2]^{17}, [-3]^{18}\}$	29
$\{[24]^{18}, [14]^9, [8]^9\}$	$\{[20]^1, [2]^{17}, [-3]^{18}\}$	240
$\{[35]^1, [26]^7, [19]^{35}\}$	$\left\{[21]^1, [\frac{-1\pm\sqrt{41}}{2}]^{21}\right\}$	949
$\{[35]^1, [26]^7, [19]^{35}\}$	$\left\{[21]^1, [\frac{-1\pm\sqrt{41}}{2}]^{21}\right\}$	1849

Bridges and Mena, Aequationes Math. (1981)
Van Dam, JCTB (1998); De Caen et al., JCTA (1999)
Cheng et al., European J. Combin. (2016)

Graphs with three valencies

Theorem (GG and Yip 2023+)
Let Γ be connected w/ three distinct eigenvalues and three distinct valencies. Then $\mathrm{rank}(\mathcal{W}(\Gamma)) \geqslant 14$.

$\mathcal{Q}$: quasi-symmetric 2-$(85, 35, 34)$ design with intersection numbers 10 and 15.

???

Does $\mathcal{Q}$ exist?

Γ: cone over the total graph of $\mathcal{Q}$.

Properties of Γ:

- valencies $\{[289]^1, [169]^{85}, [64]^{204}\}$;
- spectrum $\{[119]^1, [4]^{204}, [-11]^{85}\}$;
- coherent rank 14.

DRACK$_n$ and LSSD

Van Dam 1998: infinite family of graphs w/ three distinct eigenvalues and coherent rank 10.

$$\begin{bmatrix} J-I & B & B & B & B \\ B^\intercal & & & & \\ B^\intercal & & \multicolumn{3}{c}{(4t^2,4,t^2)\text{-}} \\ B^\intercal & & \multicolumn{3}{c}{\text{DRACK}_n} \\ B^\intercal & & & & \end{bmatrix}$$

B: incidence matrix of symmetric 2-$(4t^2, 2t^2-t, t^2-t)$ design.

Known to exist when $t = 2^i$ with $i \geqslant 1$.

Van Dam 1998: another rank-10 construction from a Linked System of Symmetric Designs.

Both constructions have type $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 4 \end{smallmatrix}\right]$. Is type $\left[\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right]$ possible?

More questions

Question.
Do there exist connected graphs with three distinct eigenvalues and coherent rank 11?

- Infinite families known for ranks 3, 5, 6, 8, 9, 10.

Question (De Caen 1999).
Does a connected graph with three distinct eigenvalues have at most three distinct valencies?

- Cheng et al. (2016): Yes, when complement is disconnected.
- Van Dam et al. (2015): Connected graphs with four distinct eigenvalues can have arbitrarily many distinct valencies.

Thanks!