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Transitive path decompositions of Cartesian products of complete graphs

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joint work with

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Definition (H-decomposition)

Let H be a graph, an *H-decomposition* of a graph $\Gamma = (V, E)$ is a collection $\mathscr{D}$ of edge-disjoint subgraphs of Γ, each isomorphic to H, whose edge sets partition the edge set E of Γ.

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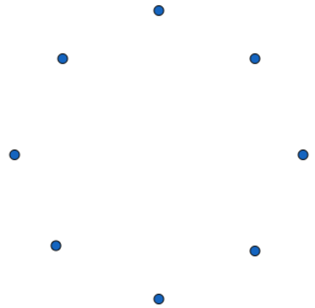

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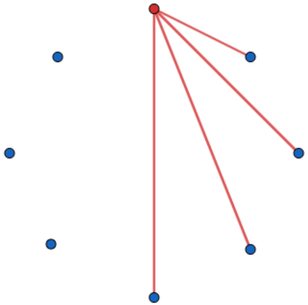

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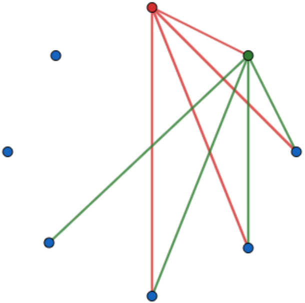

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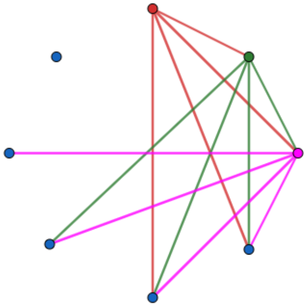

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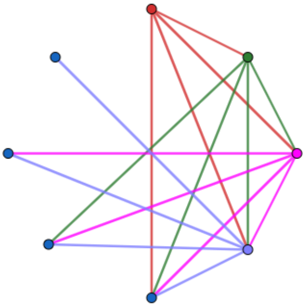

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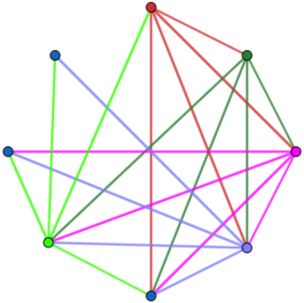

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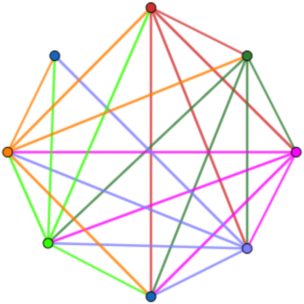

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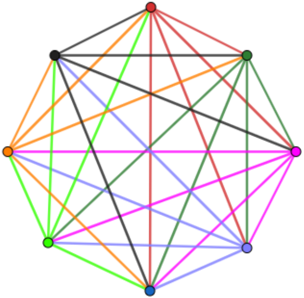


Definition (Automorphism of a graph)

An *automorphism* of Γ is a permutation of the vertex set V of Γ which leaves the edge set E of Γ invariant.

Definition (G-transitive decomposition)

Let G be an automorphism group of the graph Γ. We say that the H-decomposition $\mathscr{D}$ of Γ is G-transitive if the following two conditions hold.

1. G leaves $\mathscr{D}$ invariant, that is for all $H \in \mathscr{D}$ and $g \in G$, we have $H^g \in \mathscr{D}$.
2. G acts transitively on $\mathscr{D}$, that is for any $H_1, H_2 \in \mathscr{D}$, there exists a $g \in G$ such that $H_1^g = H_2$.

If these conditions hold then we call the triple $(G, \Gamma, \mathscr{D})$ a *transitive H-decomposition*.

Theorem (D and Devillers 2023+)

Let Γ be a graph and H be a subgraph of Γ. Suppose that $G \leqslant \mathrm{Aut}(\Gamma)$ is semiregular on the edges of Γ and H contains exactly one edge from each edge orbit of Γ under G. Then H^G is a G-transitive H-decomposition of Γ of size $|G|$.

Example

Let $\Gamma = K_9$ and we denote the vertex set of Γ by $\{1, 2, 3, \dots, 9\}$. Let $G = \langle (1,4,7)(2,5,8)(3,6,9) \rangle$ of order 3. We have listed the edge orbits of K_9 under G and highlighted the edges of H.

Orbit 1:	$[\{\mathbf{1,4}\},\{4,7\},\{7,1\}]$	Orbit 2:	$[\{2,5\},\{5,8\},\{\mathbf{8,2}\}]$
Orbit 3:	$[\{3,6\},\{6,9\},\{\mathbf{9,3}\}]$	Orbit 4:	$[\{1,2\},\{\mathbf{4,5}\},\{7,8\}]$
Orbit 5:	$[\{\mathbf{1,5}\},\{4,8\},\{7,2\}]$	Orbit 6:	$[\{1,6\},\{4,9\},\{\mathbf{7,3}\}]$
Orbit 7:	$[\{2,3\},\{\mathbf{5,6}\},\{8,9\}]$	Orbit 8:	$[\{2,4\},\{\mathbf{5,7}\},\{8,1\}]$
Orbit 9:	$[\{\mathbf{2,6}\},\{5,9\},\{8,3\}]$	Orbit 10:	$[\{\mathbf{3,7}\},\{6,1\},\{9,4\}]$
Orbit 11:	$[\{3,5\},\{\mathbf{6,8}\},\{9,2\}]$	Orbit 12:	$[\{\mathbf{3,9}\},\{6,3\},\{9,6\}]$

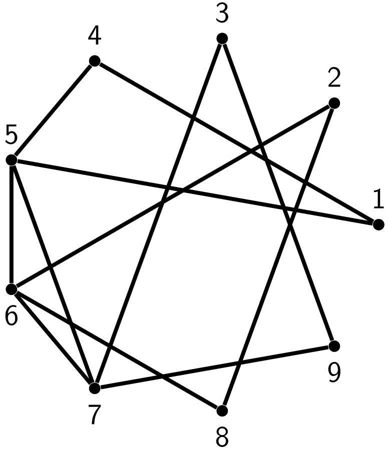


Figure: H

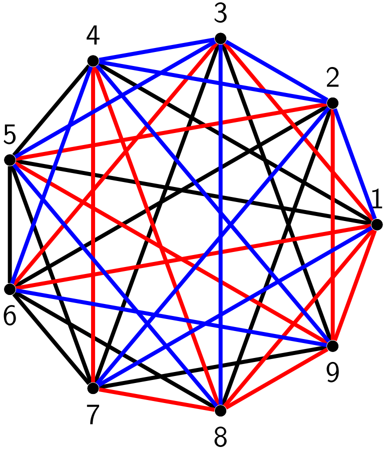


Figure: G-transitive H-decomposition of K_9

Theorem (D and Devillers 2023+)

A transitive $n(n-1)$-path decomposition of $K_n \square K_n$ exists for all odd primes n.

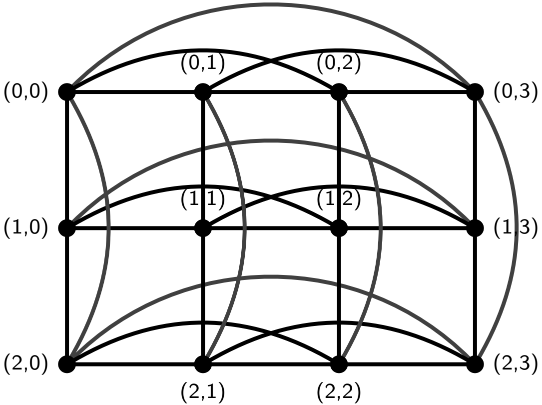
Definition (Cartesian product of complete graphs)

Let $\Gamma = K_n \Box K_m$ be the *Cartesian product* of the complete graphs K_n and K_m. The graph Γ may be viewed as a 2-dimensional 'grid' consisting of 'horizontal' copies of K_m and 'vertical' copies of K_n.

- Let $V(\Gamma)$ and $E(\Gamma)$ be the vertex set and edge set of Γ respectively.
- Let $\mathbb{Z}_n$ be the additive group of integers modulo n.
- We label the vertices of Γ as ordered pairs $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_m$.
- We define the edges of Γ as follows: for any $(a, b), (c, d) \in V(\Gamma)$, $\{(a, b), (c, d)\} \in E(\Gamma)$ whenever $a = c$ or $b = d$.

Example

$\Gamma = K_3 \square K_4$ where each horizontal line induces a K_4 and each vertical line induces a K_3.



Definition (Array representation of a walk)

Let $W = v_0 v_1 v_2 \dots v_{\ell-1} v_\ell$ be an ℓ-walk in Γ.

For a given walk W with fixed v_0, we can define an array $\overrightarrow{a} = [a_1, a_2, \dots, a_\ell]$ such that $a_i = v_i - v_{i-1}$ for $1 \leqslant i \leqslant \ell$ and $|\overrightarrow{a}| = \ell$.

This implies $a_i \in \{(c, 0), (0, c') | c \in \mathbb{Z}_n^* \text{ and } c' \in \mathbb{Z}_m^*\}$.

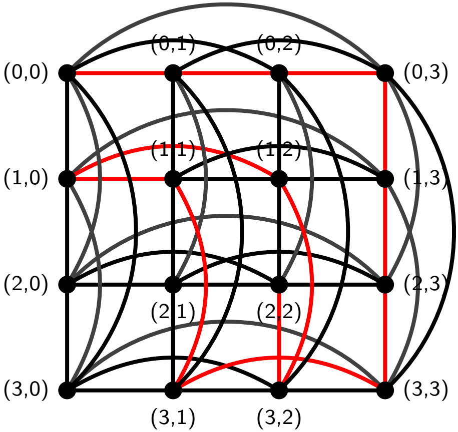
Conversely, a given v_0 and array $\overrightarrow{a}$ determines W since $v_i = v_0 + \sum_{g=1}^{i} a_g$.

We write $W = W(v_0, \overrightarrow{a})$.

If $W(v_0, \overrightarrow{a})$ is a path, then we denote it by $P(v_0, \overrightarrow{a})$.

Example

A 12-path $P = P((0,0), \overrightarrow{a})$ where
$\overrightarrow{a} = [01, 01, 01, 10, 10, 10, 02, 20, 03, 02, 20, 30]$.



Lemma

An ℓ-walk $W = W(v_0, \overrightarrow{a})$ is an ℓ-path if and only if $\sum_{g=i+1}^{j} a_g \neq (0,0)$ for all $0 \leqslant i < j \leqslant \ell$.

Proof.

An ℓ-walk is an ℓ-path if and only if $v_i \neq v_j$ whenever $i < j$ for all $v_i, v_j \in V(W)$. That is an ℓ-walk is an ℓ-path if and only if $v_0 + \sum_{g=1}^{i} a_g \neq v_0 + \sum_{g=1}^{j} a_g$, simplified to $\sum_{g=i+1}^{j} a_g \neq (0,0)$, for all $0 \leqslant i < j \leqslant \ell$.

□

Definition (Group - G)

- Consider the cyclic permutation of $V(\Gamma)$ defined by $c(a,b) = (a+1,b)$ for all $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_m$.
- Let $G = \langle c \rangle$, which permutes the rows cyclically.
- This implies that the order of G is n.

Definition (Horizontal edge orbits)

Each horizontal edge orbit under G can be denoted as the set $\{\{(a,b_1),(a,b_2)\} \in E(\Gamma) | a \in \mathbb{Z}_n\}$ for fixed $b_1 \neq b_2 \in \mathbb{Z}_m$.

Definition (Vertical edge orbits)

Each vertical edge orbit under G can be denoted as the set $\{\{(a,b),(a+t,b)\} \in E(\Gamma) | a \in \mathbb{Z}_n\}$ for fixed $t \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_m$.

Lemma

Let n be an odd integer and $\Gamma = K_n \Box K_m$.

(a) *There are $\binom{m}{2}$ horizontal edge orbits of size n under G.*

(b) *There are $\frac{m(n-1)}{2}$ vertical edge orbits of size n under G.*

By the orbit-stabilizer theorem, it follows that the stabilizer G_e of e under the group G is trivial thus we have the following corollary.

Corollary

Let n be an odd integer and $\Gamma = K_n \Box K_m$. Then G acts semiregularly on $E(\Gamma)$.

We denote the first coordinate and second coordinate of an ordered pair v by v_1 and v_2 respectively. If $v = (a, b)$, then $v_1 = a$ and $v_2 = b$.

Lemma

Any two distinct edges $\{v, w\}$ and $\{v', w'\}$ of $E(\Gamma)$ are in the same edge orbit under G if and only if

(a) $v_2 = v_2'$ *and* $w - v = w' - v'$ *or,*

(b) $v_2 = w_2'$ *and* $w - v = v' - w'$.

Lemma

Consider an ℓ-path $P = P(v_0, \overrightarrow{a})$ in Γ. For $0 \leqslant i < j < \ell$ we define the following conditions:

(a) $(\sum_{g=i+1}^{j} a_g)_2 \neq 0$ *or* $a_{i+1} \neq a_{j+1}$,

(b) $(\sum_{g=i+1}^{j+1} a_g)_2 \neq 0$ *or* $a_{i+1} \neq -a_{j+1}$.

Then P contains at most one edge from each edge orbit under G if and only if (a) and (b) hold for all $i, j \in \{0, 1, \ldots, \ell - 1\}$ with $i < j$.

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Example

A 6-path $P = P((0,0), \overrightarrow{a})$ in $K_3 \Box K_3$ with
$\overrightarrow{a} = [(0,1),(1,0),(0,1),(1,0),(0,1),(2,0)]$.

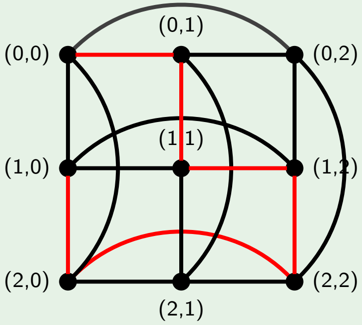


Figure: The 6-path is represented by the red edges.

Example

$\overrightarrow{a} = [(0,1),(1,0),(0,1),(1,0),(0,1),(2,0)]$.

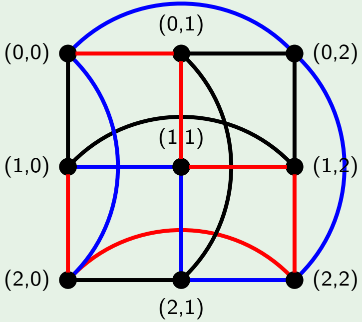
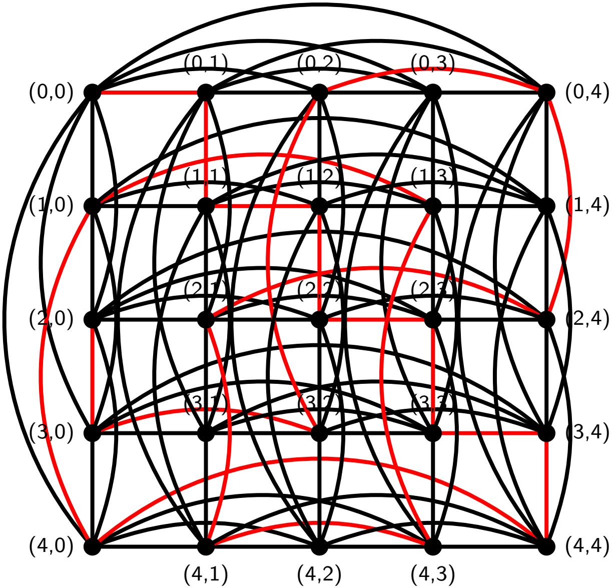


Figure: $P = P((0,0),\overrightarrow{a})$ in red, $P' = P^c = P((1,0),\overrightarrow{a})$ in blue and $P'' = P^{c^2} = P((2,0),\overrightarrow{a})$ in black.

Example

A 20-path $P = P((0,0), \overrightarrow{a})$ in $K_5 \square K_5$ with
$\overrightarrow{a} = [(0,1),(1,0),(0,1),(1,0),(0,1),(1,0),(0,1),(1,0),(0,1),(2,0),$
$(0,3),(3,0),(0,3),(3,0),(0,3),(3,0),(0,3),(3,0),(0,3),(4,0)]$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(4,0)
(4,1)
(4,2)
(4,3)
(4,4)

1. **Staircase array**: For an odd integer n, we refer to an array $\overrightarrow{a}$ in the form of

$$[(0,1),(1,0),\ldots,(0,1),(2,0),(0,3),(3,0),\ldots,(0,3),(4,0),\\ \ldots,(0,n-2),(n-2,0)\ldots,(0,n-2),(n-1,0)]$$

as the *staircase array*.

2. Proved that when n is an odd prime, $v_0=(0,0)$ and $\overrightarrow{a}$ is the staircase array, $P=P(v_0,\overrightarrow{a})$ represents an $n(n-1)$-path in $K_n\square K_n$.

3. Moreover, proved that P contains exactly one edge from each edge orbit.

4. G is semiregular on $E(K_n\square K_n)$, therefore P^G is a G-transitive $n(n-1)$-path decomposition of $K_n\square K_n$.

- We were mainly focusing on the group $G = \langle c \rangle$ where $c(a,b) = (a+1,b)$ for all $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_m$. In this case, we need n to be odd to maintain the semiregular action on the edges.

- We also found L-transitive $n(n-1)$-path decompositions of $\mathbb{Z}_n \times \mathbb{Z}_n$ when $n = 2,3,4$, for $L = \langle c' \rangle$ where $c'(a,b) = (a+1,b+1)$ for all $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

For example, for $n = 4$, we take $H = P((0,0), \overrightarrow{a})$ where

$$\begin{aligned}\overrightarrow{a} =&[(0,1),(0,1),(0,1),(1,0),(1,0),(1,0),\\&(0,2),(2,0),(0,3),(2,0),(0,2),(3,0)]\end{aligned}$$

and decomposition H^L.

- Another interesting question would be considering $K_n \square K_m$ for distinct n and m (rectangular grid), and various possible subgraphs, not just a path.