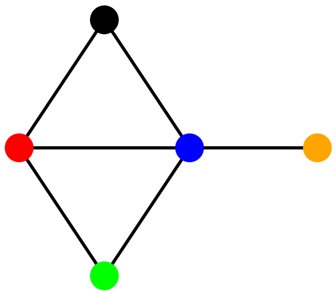
Algebraic Graph Theory and Quantum Walks

Krystal Guo

University of Amsterdam
Korteweg de Vries Institute for Mathematics

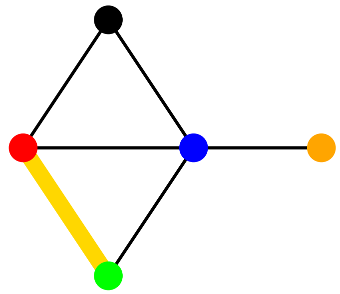
45ACC, University of Western Australia, Perth, Dec 12, 2023.

Algebraic Graph Theory



a graph G

Algebraic Graph Theory



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adjacency matrix

$$A := A(G)$$

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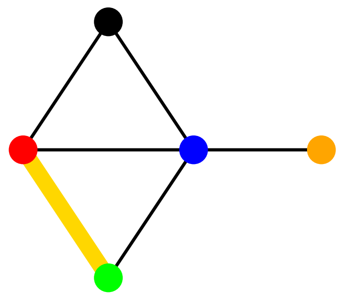

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$$
\begin{array}{c c}
 & \begin{matrix} \color{black}{\bullet} & \color{red}{\bullet} & \color{green}{\bullet} & \color{blue}{\bullet} & \color{orange}{\bullet} \end{matrix} \\
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spectral bounds

cospectrality

matrix algebras

etc.

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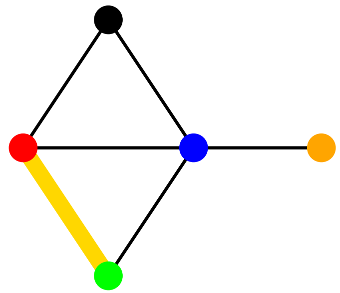

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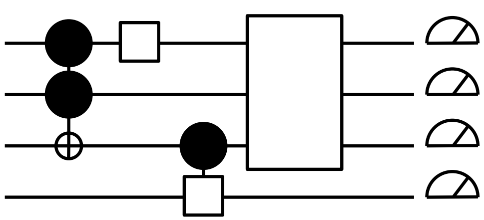
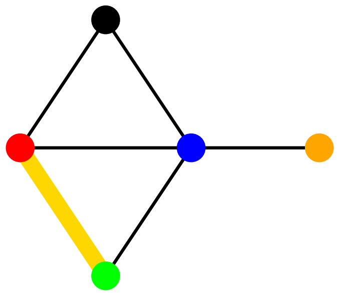


quantum circuit

Algebraic Graph Theory



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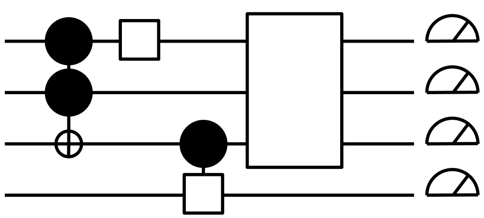

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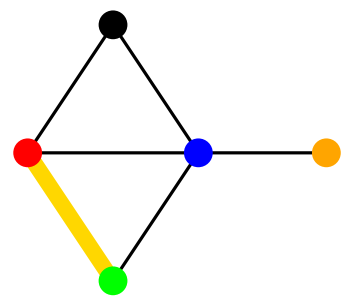

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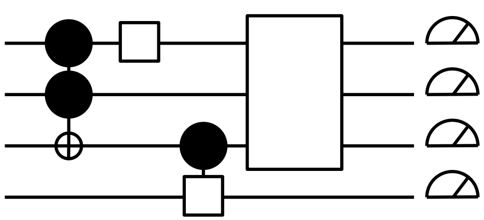

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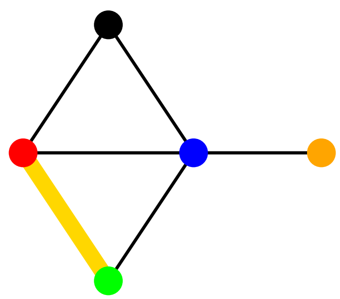
UvA



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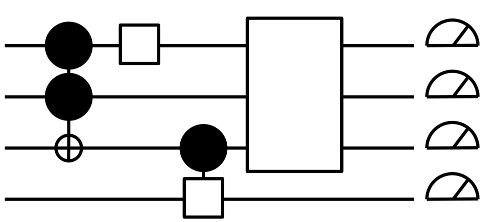

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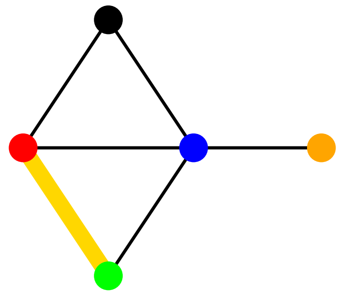

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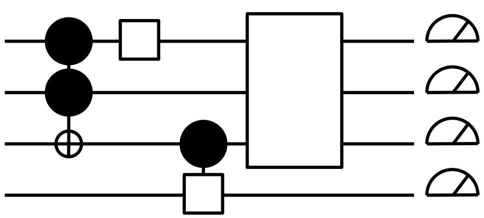

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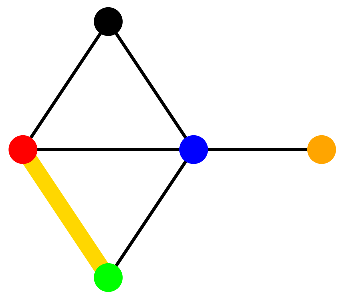

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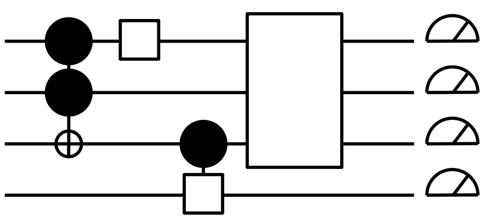

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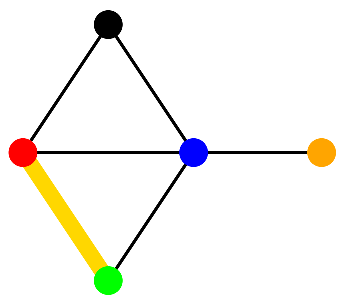

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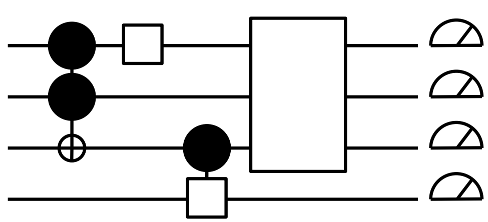

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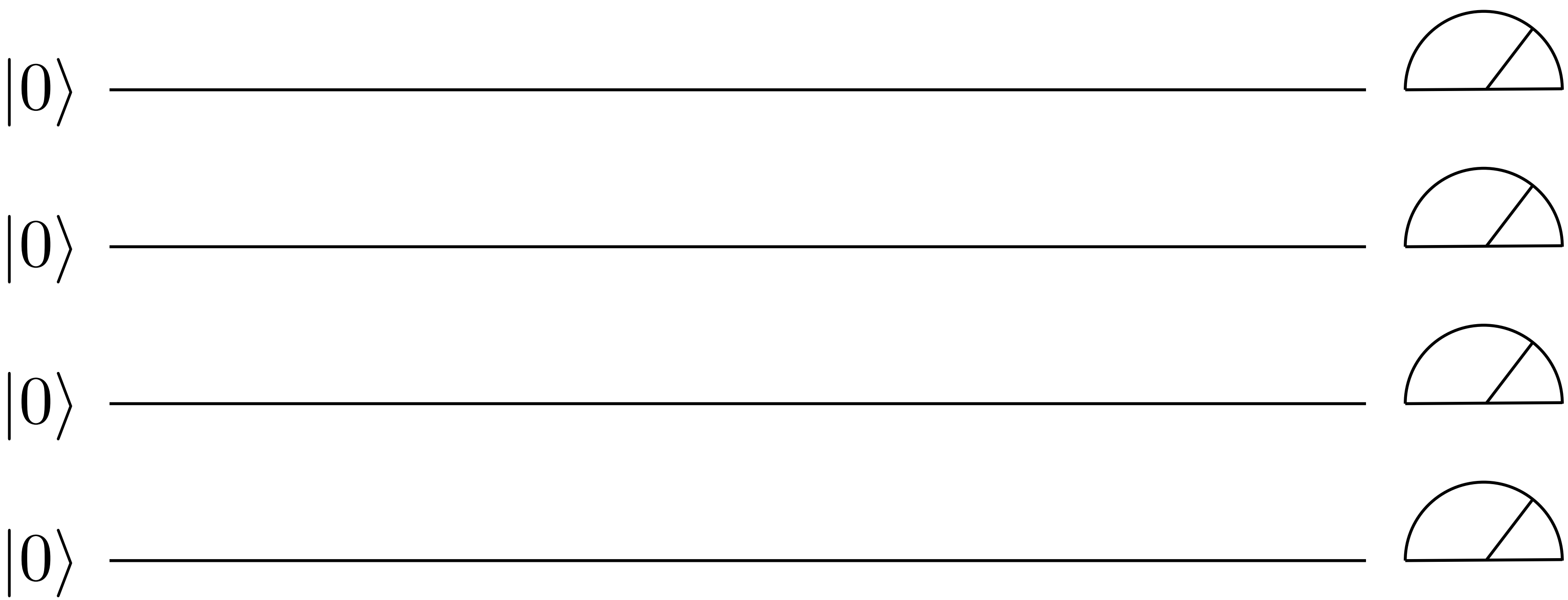
Continuous-time Quantum Walks

Quantum computing



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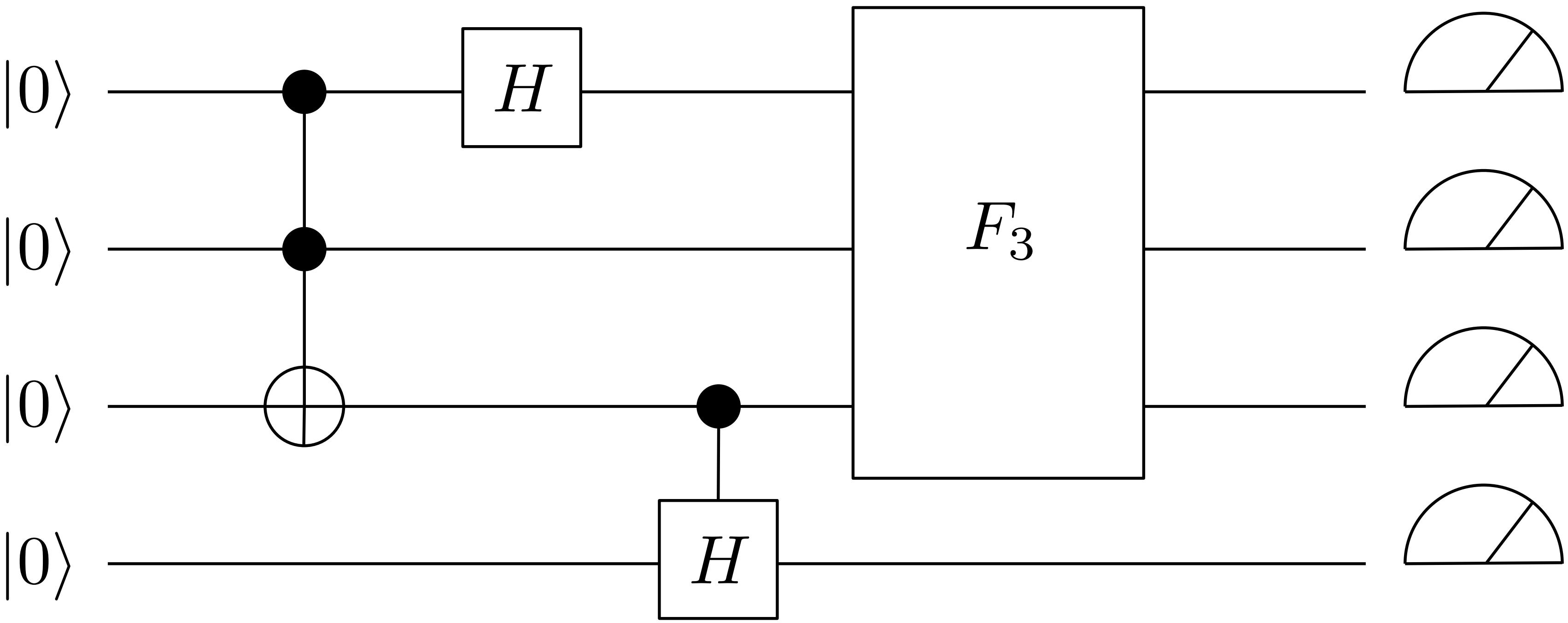

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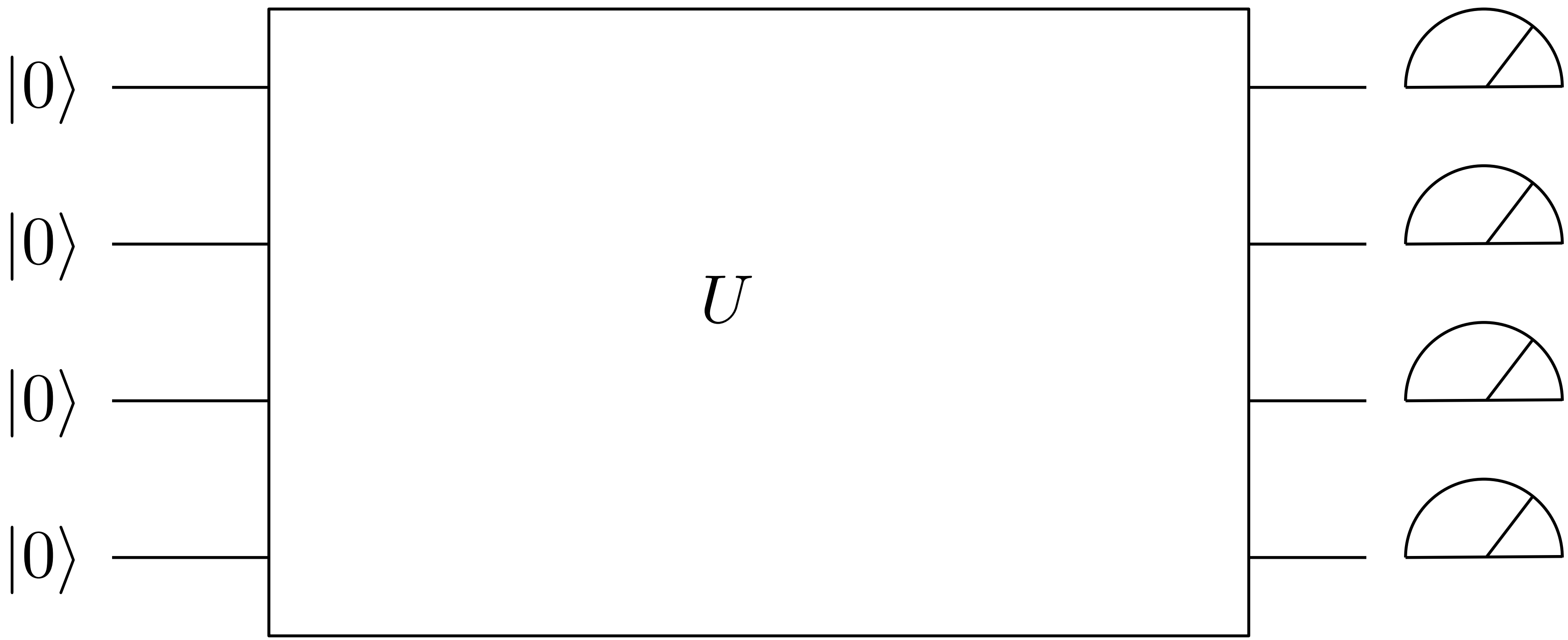

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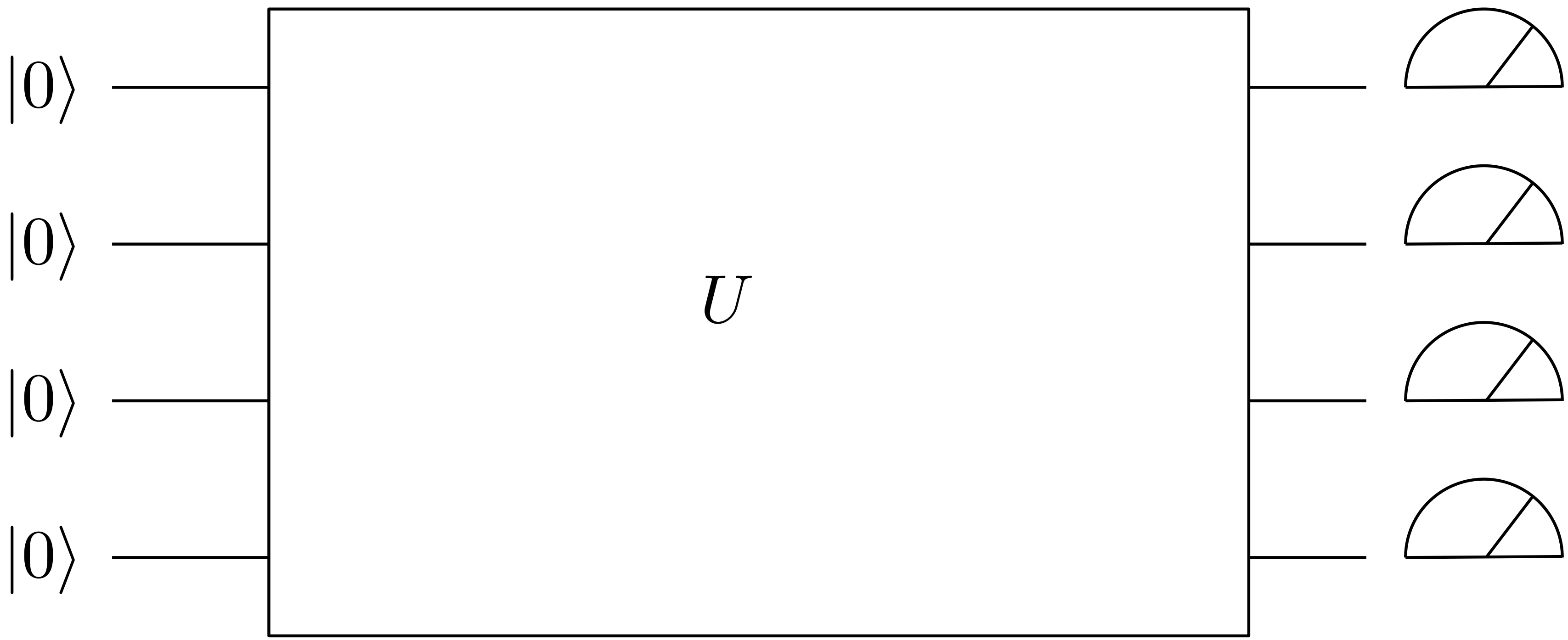

With some assumptions about the system, we can model it by a $n \times n$ matrix, $U(t) = e^{itA}$ where A is the adjacency matrix of an underlying graph.

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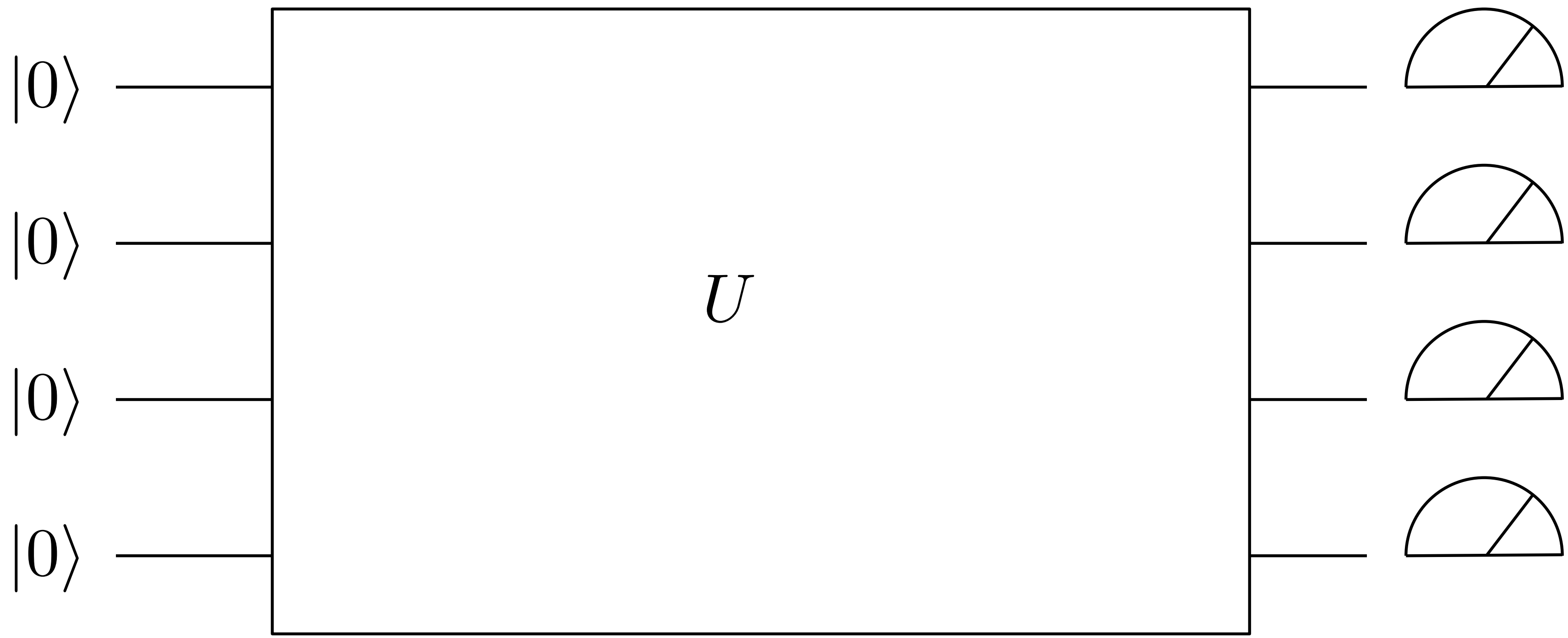

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XY-Hamiltonian

UvA

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Hamiltonian of graph G: $H_{xy} = \frac{1}{2} \sum_{uv \in E} (\sigma_x^u \sigma_x^v + \sigma_y^u \sigma_y^v)$

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Continuous-time quantum walk

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Transition matrix

$$U(t) = \exp(itA)$$

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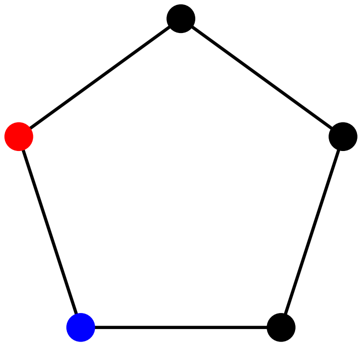
$$
\begin{aligned}
U(t) &= \exp(itA) \\
&= I + itA - \tfrac{1}{2!}t^2A^2 - \tfrac{i}{3!}t^3A^3 + \cdots
\end{aligned}
$$

Example

$U(t)_{a,b}$ for $t \in [0, 100]$.



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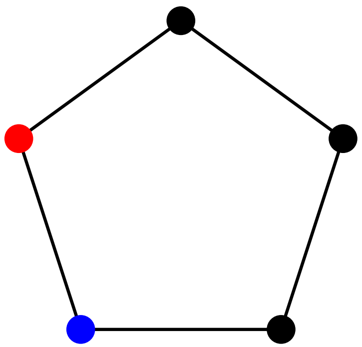
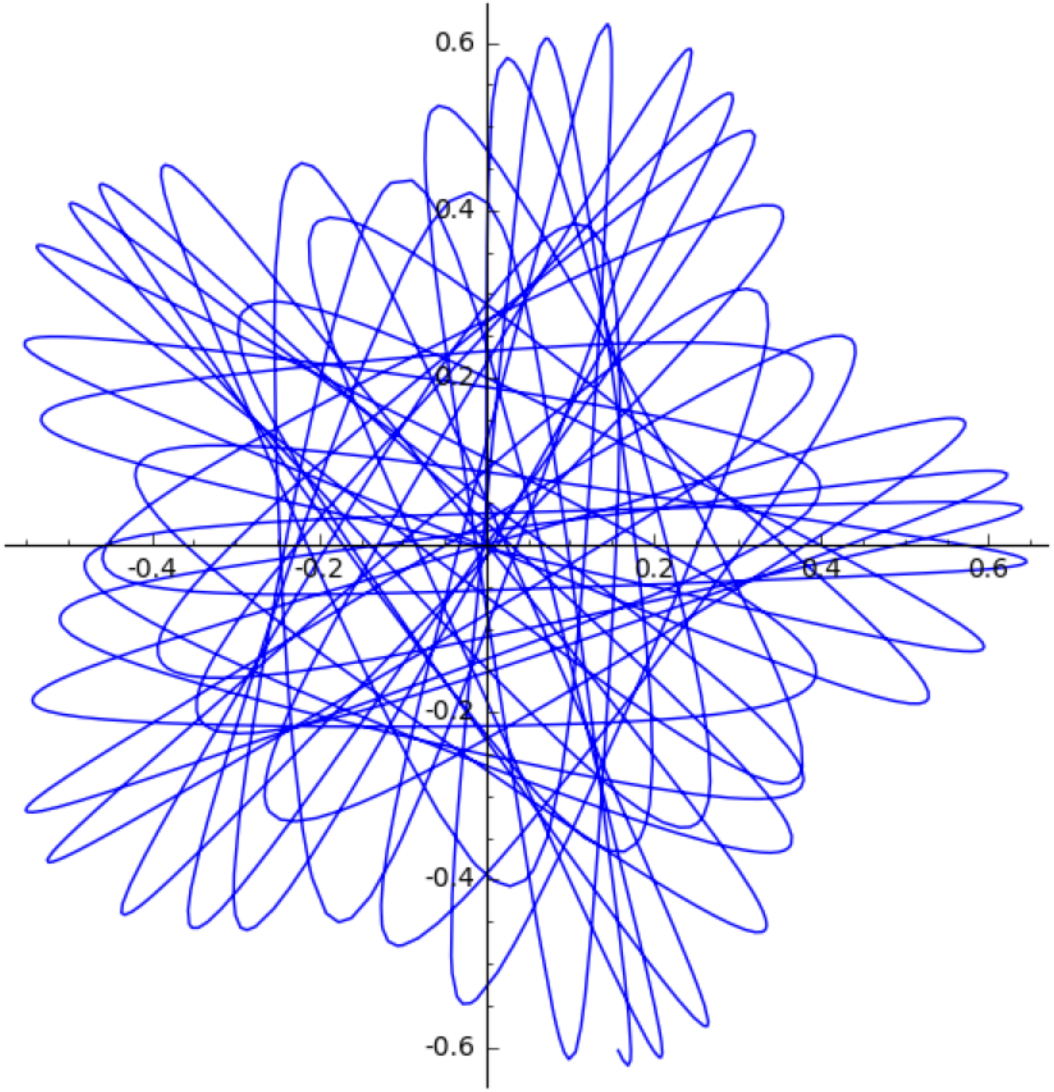
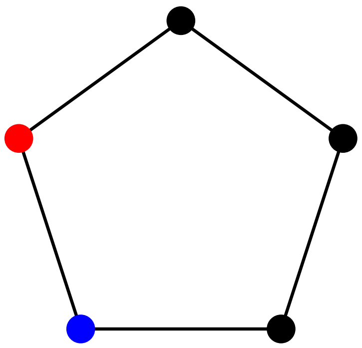




Example

$U(t)_{a,b}$ for $t \in [0, 500]$.



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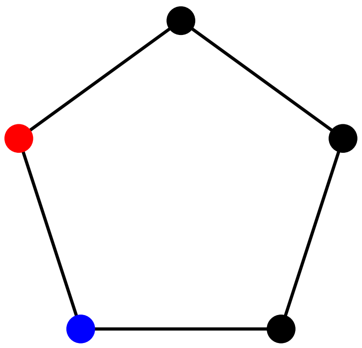

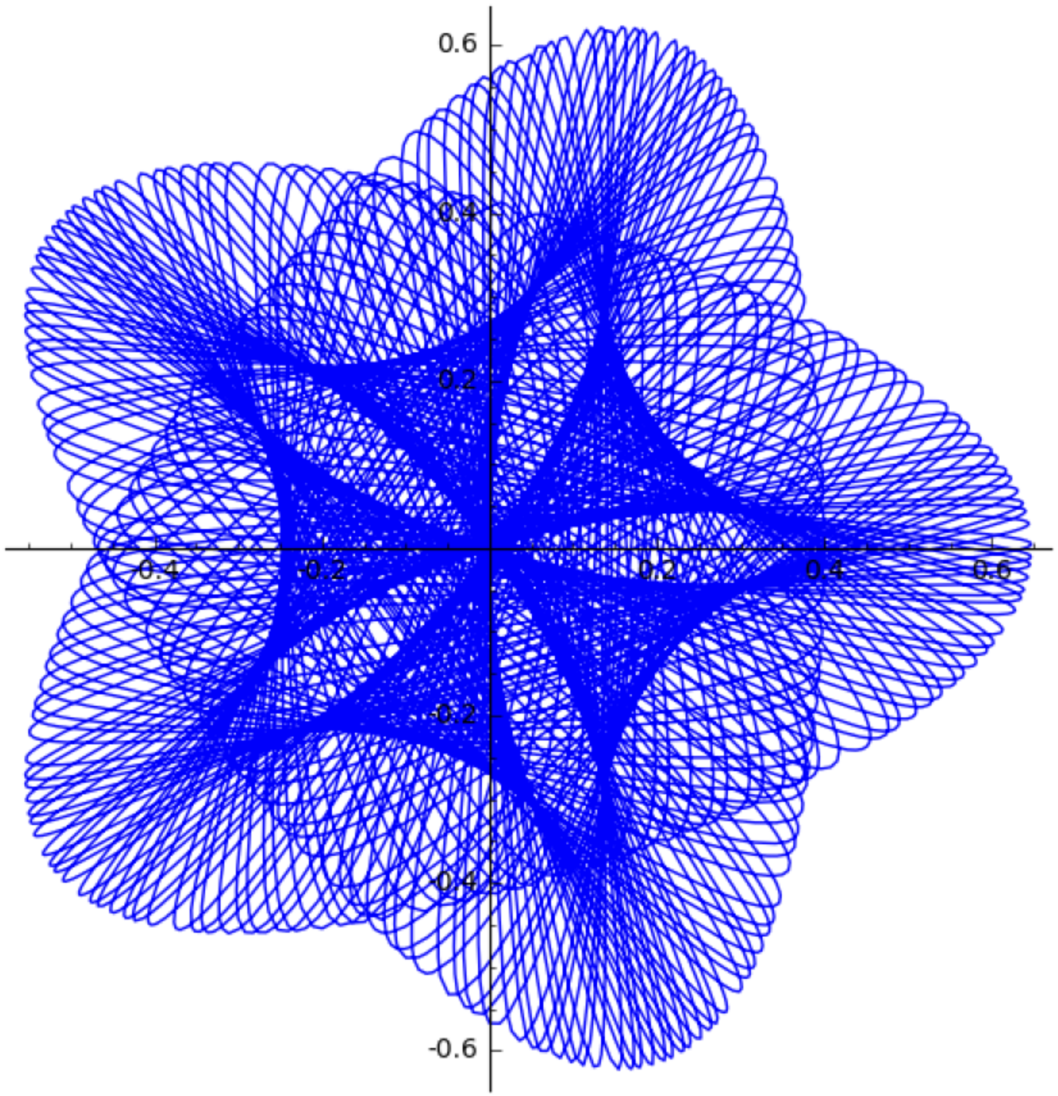

More context: quantum search

Problem: given a “marked” value, search N locations to find the location whose content is the given value.

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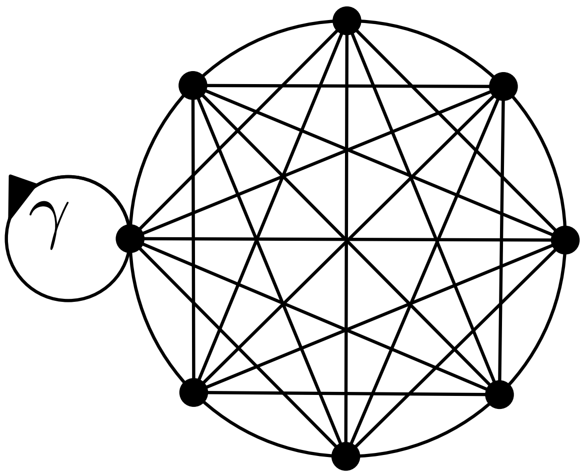

Grover's search is equivalent to running a quantum walk on K_N with a marked vertex, with the Laplacian matrix, and doing a measurement after $\sqrt{N}$ time.

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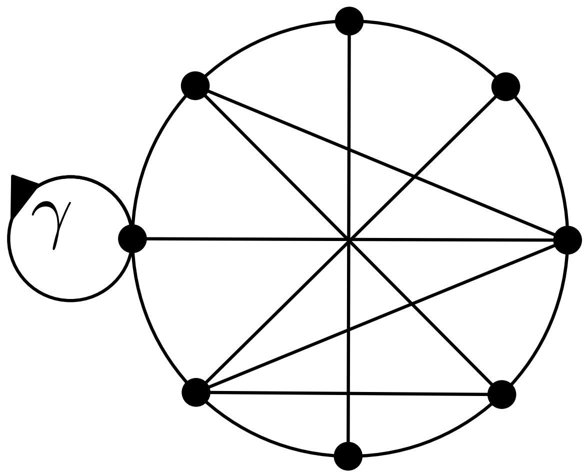
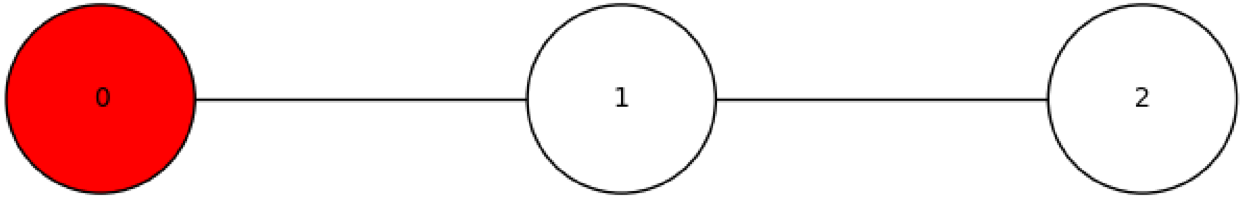
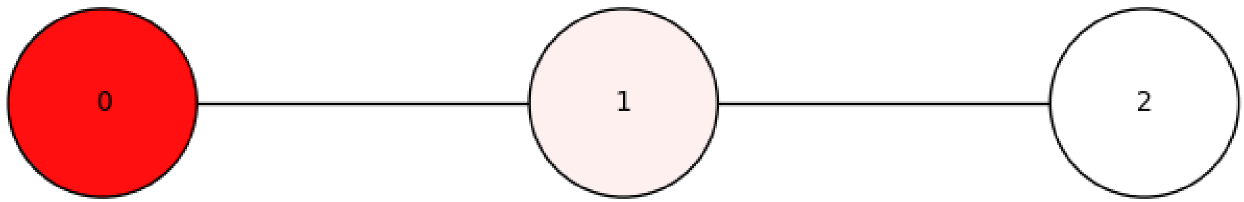


Spatial quantum search is when we run the analoguous search on a marked graph. It is not known for which graphs, spatial search has a quadratic speedup.



$$M(t) = U(t) \circ \bar{U}(t) = \begin{pmatrix} \boxed{\begin{matrix} 1.0 \\ 0.0 \\ 0.0 \end{matrix}} & \begin{matrix} 0.0 \\ 1.0 \\ 0.0 \end{matrix} & \begin{matrix} 0.0 \\ 0.0 \\ 1.0 \end{matrix} \end{pmatrix}$$

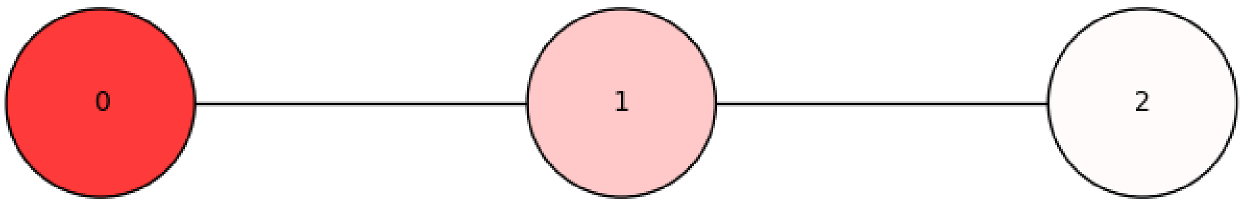
$$t = 0$$



$$\begin{pmatrix} 0.939105 & 0.059939 & 0.000956 \\ 0.059939 & 0.880122 & 0.059939 \\ 0.000956 & 0.059939 & 0.939105 \end{pmatrix}$$

$t = 0$

Time incrementing by 0.25.



$$\begin{pmatrix} 0.774615 & 0.211014 & 0.014371 \\ 0.211014 & 0.577972 & 0.211014 \\ 0.014371 & 0.211014 & 0.774615 \end{pmatrix}$$

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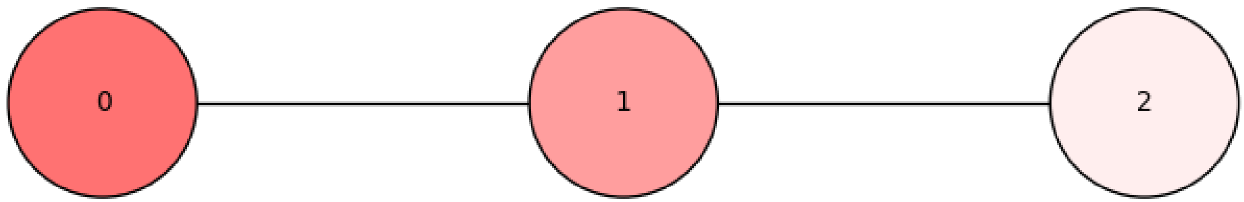

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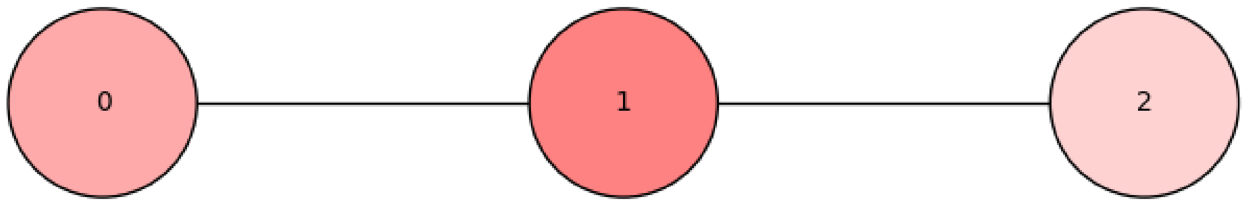

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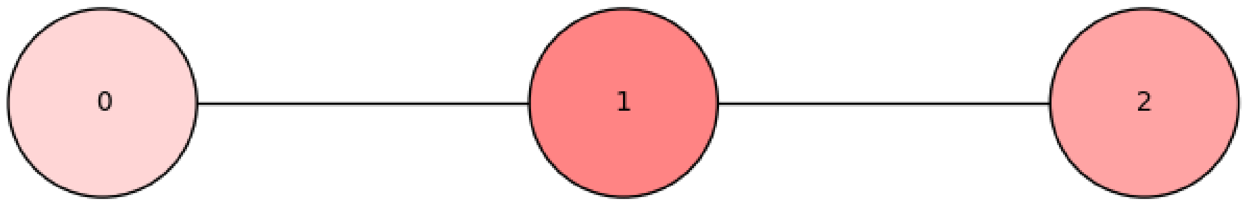

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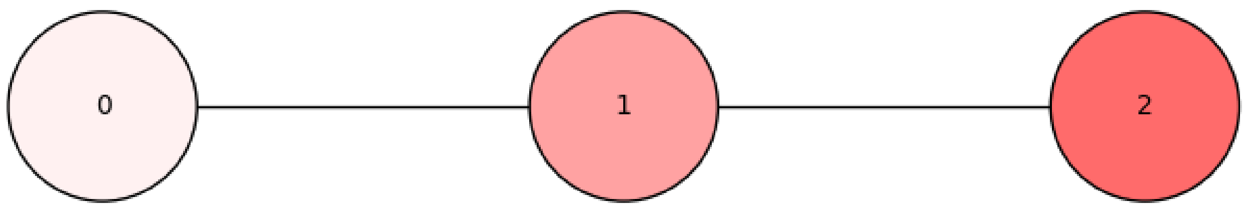

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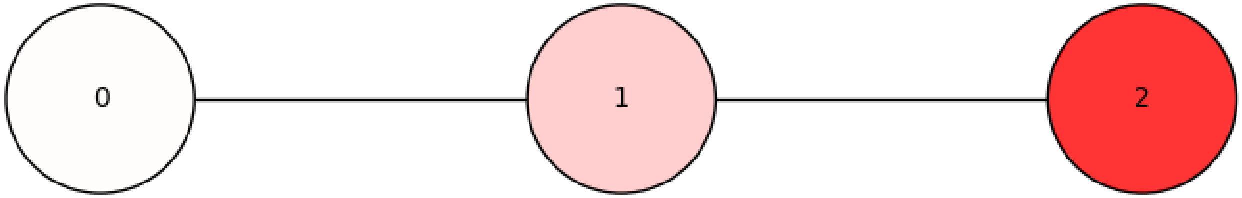

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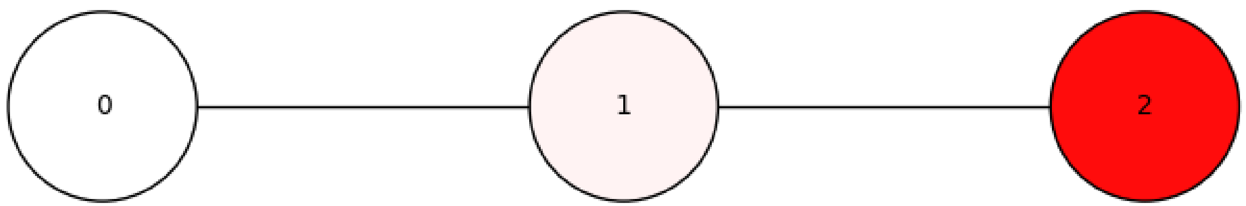

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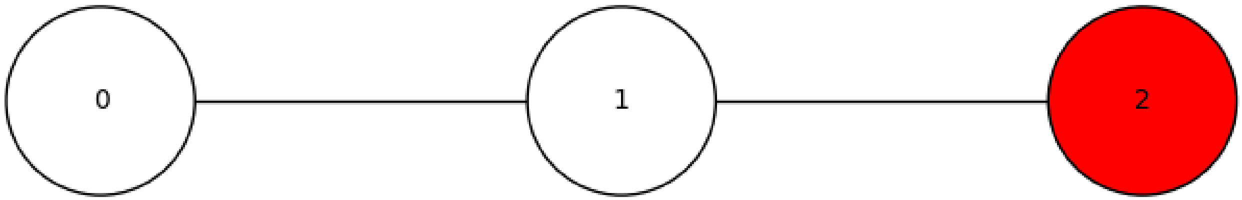

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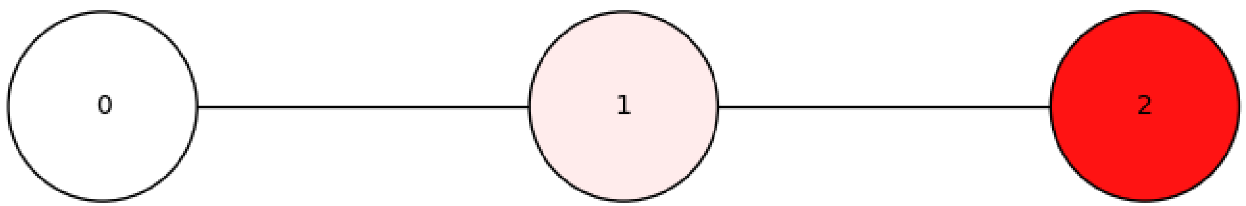

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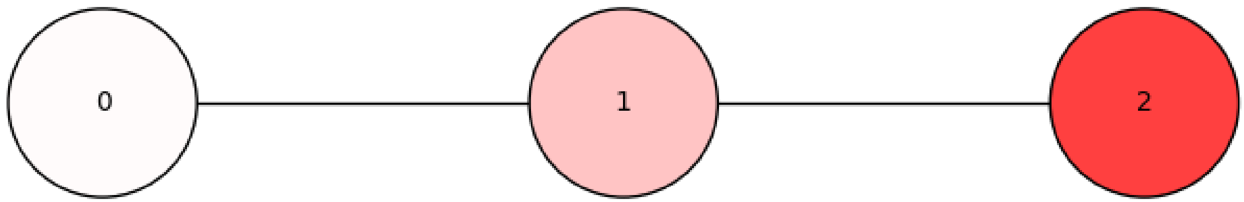

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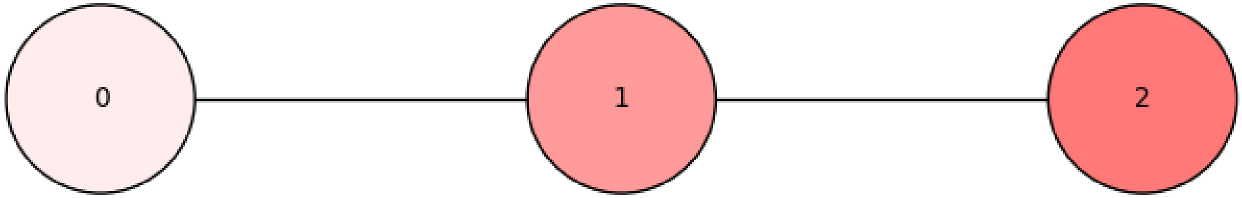

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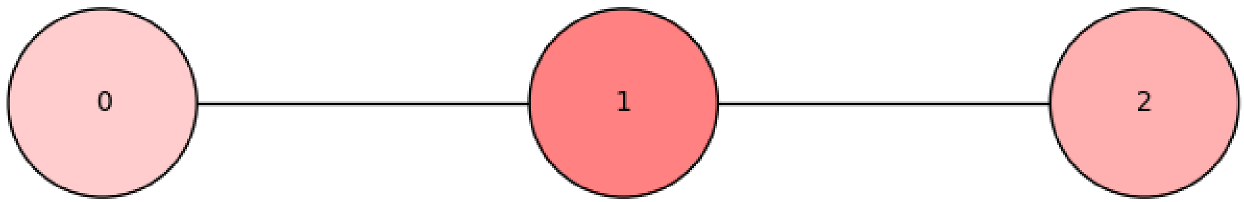

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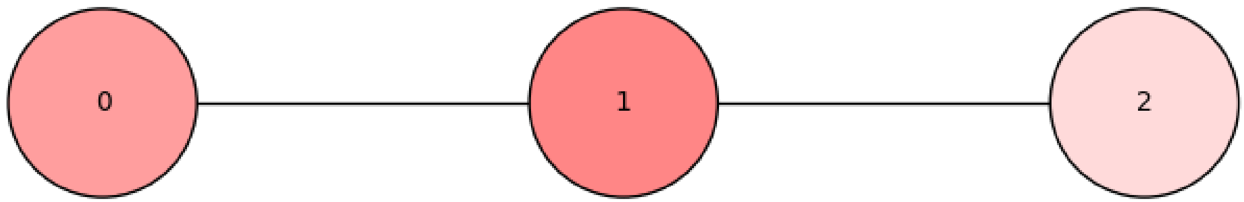

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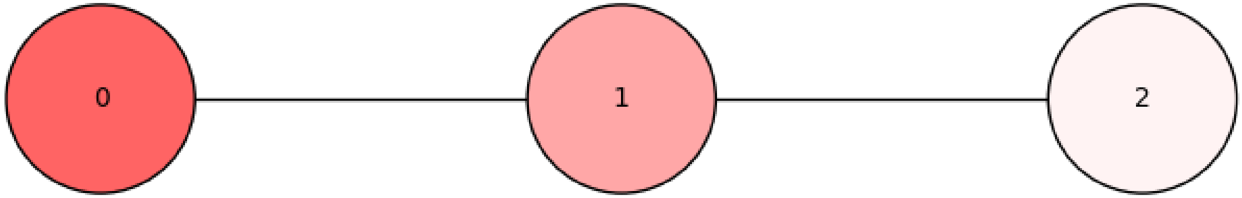

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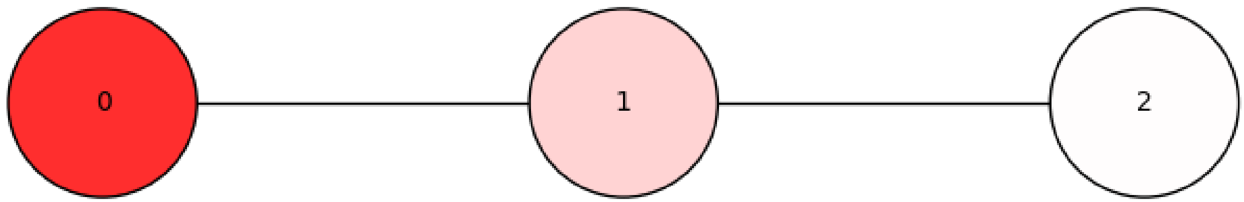

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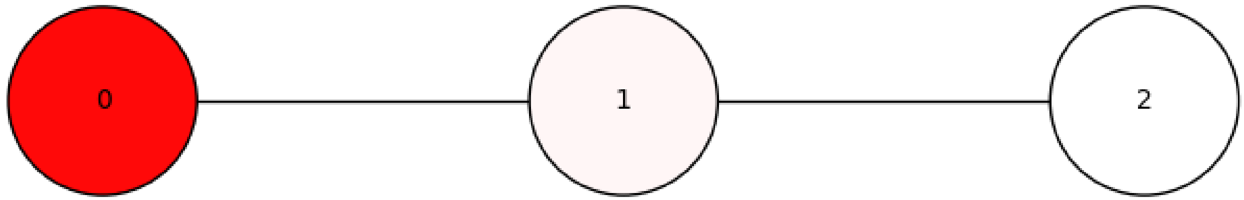

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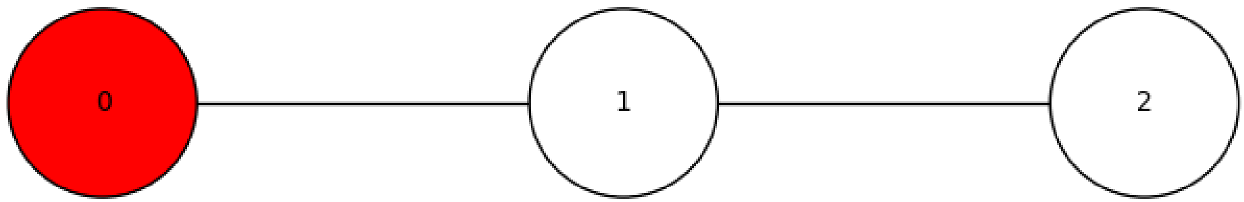

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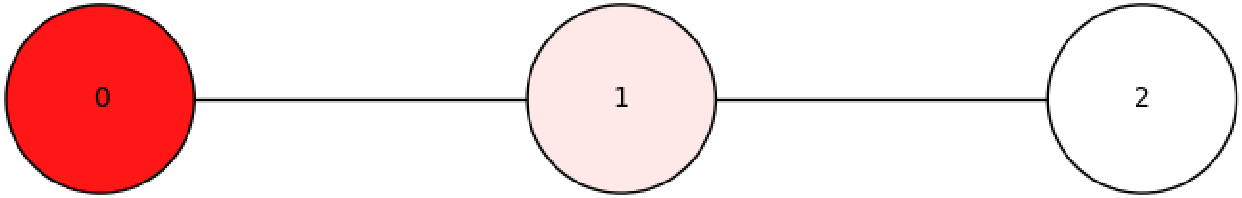

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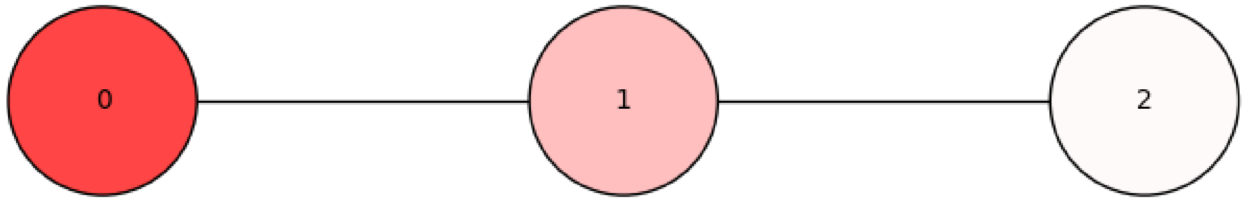

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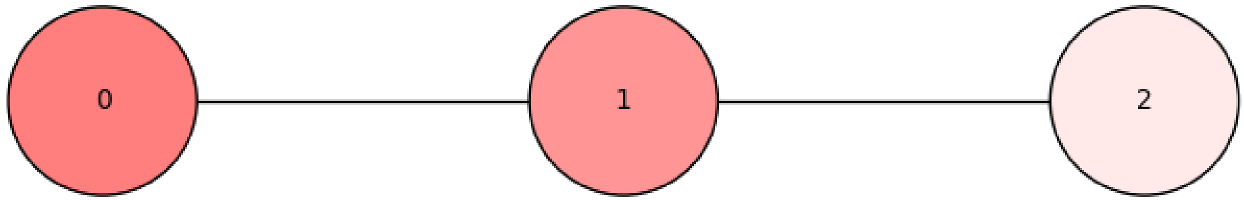

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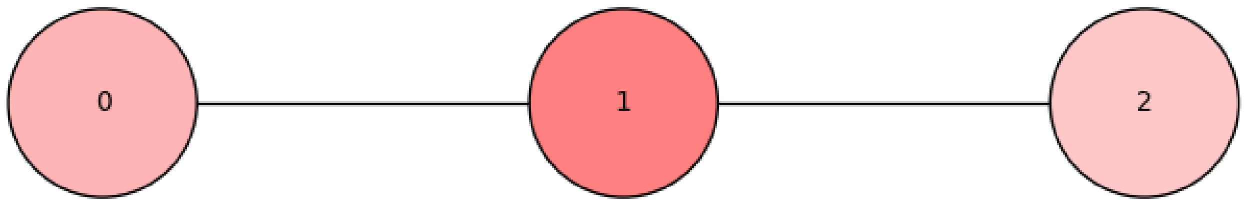

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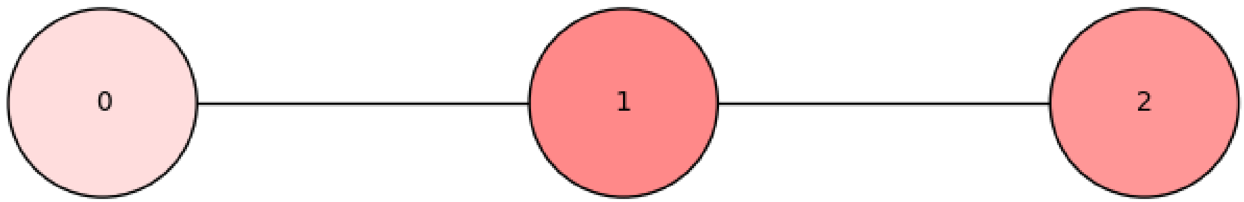

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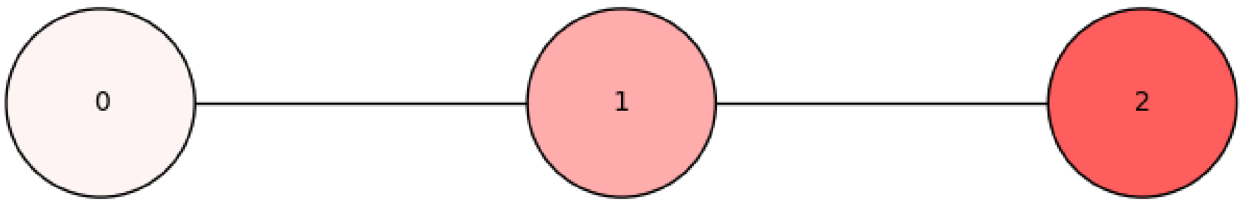

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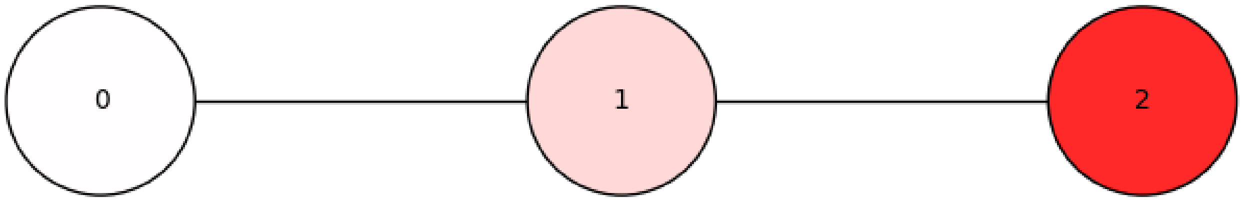

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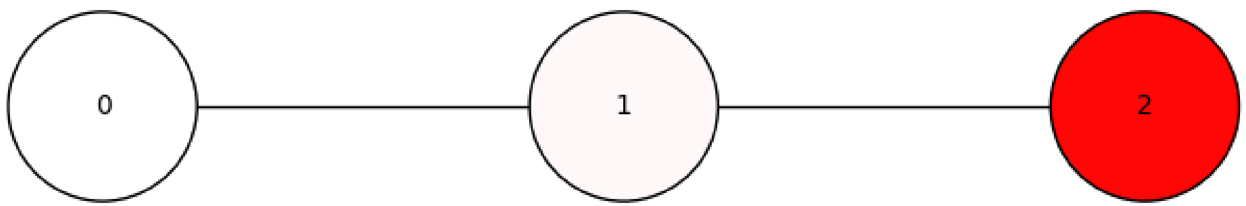

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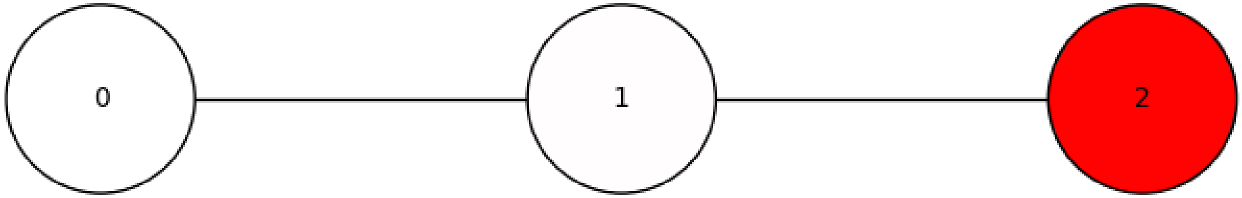

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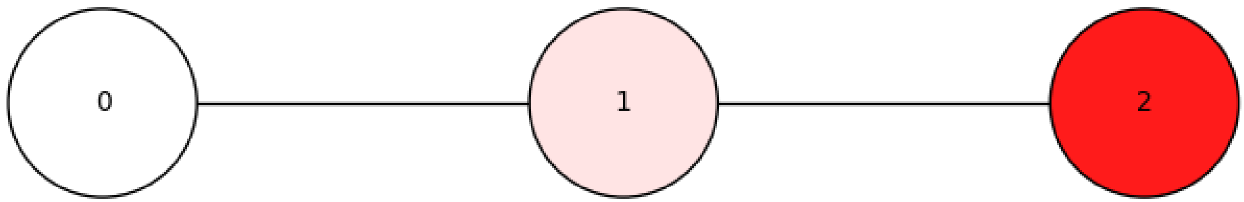

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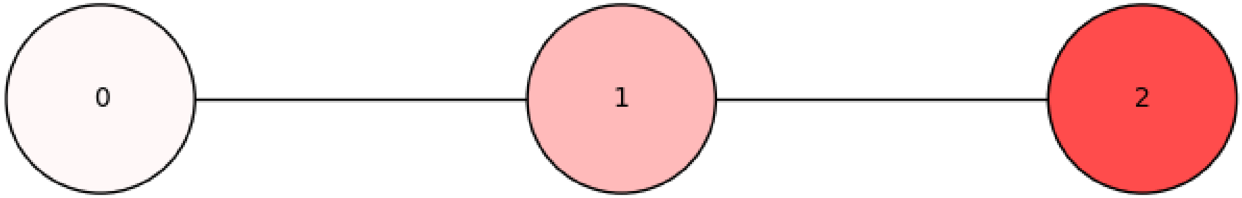

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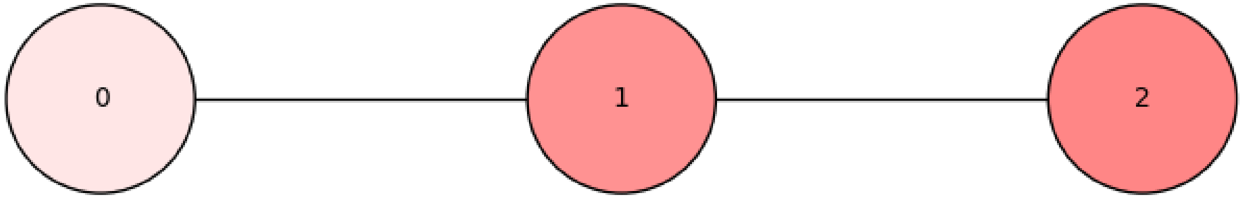

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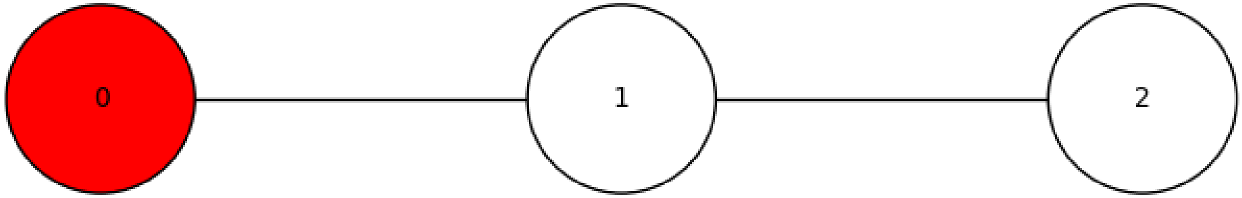
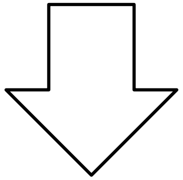
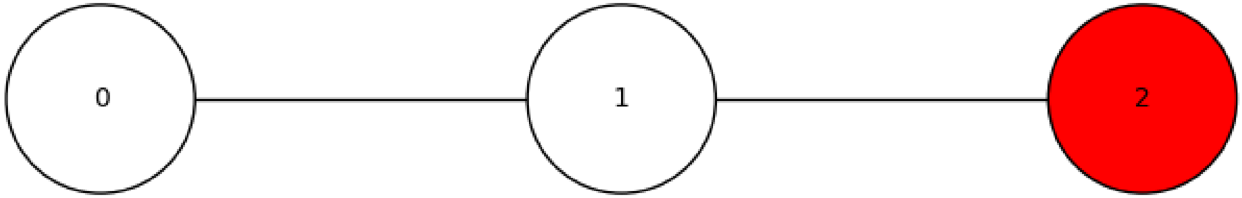
Perfect state transfer:

start





time τ



$t = 0$

Perfect state transfer: paths

Perfect state transfer from a to b:

> there exists a time τ, such that probability of measuring at b, having started at a, is 100%.

Perfect state transfer: paths

UvA

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> The only paths which admit perfect state transfer are P_2 and P_3.

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Pretty good state transfer from a to b:

> for every $\epsilon > 0$, there exists τ such that
> there exists a time τ, such that probability of measuring at b, having started at a, is at least $100 - \epsilon\%$.

Pretty good state transfer

UvA

Theorem (Godsil, Kirkland, Severini, and Smith 2012)

P_n has pretty good state transfer
if and only if $n + 1$ is a prime, twice a prime or a power of 2.

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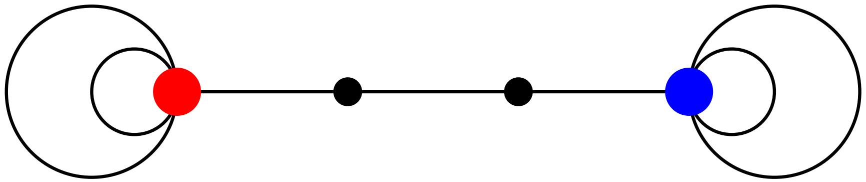
Theorem (Coutinho, Guo and van Bommel^2 2017)

P_n has pretty good state transfer between internal vxs if and only if $n+1 = 2^r p$ where p is a prime.

Perfect state transfer on paths revisited



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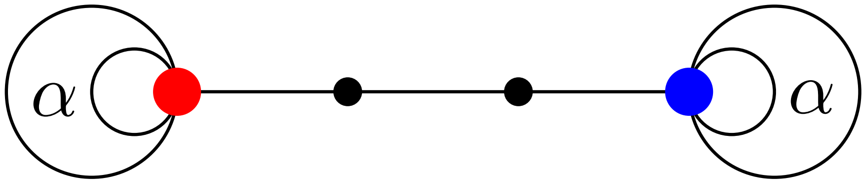

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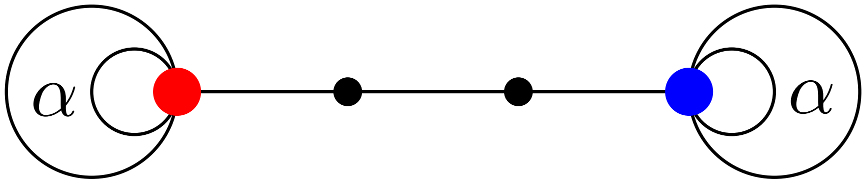
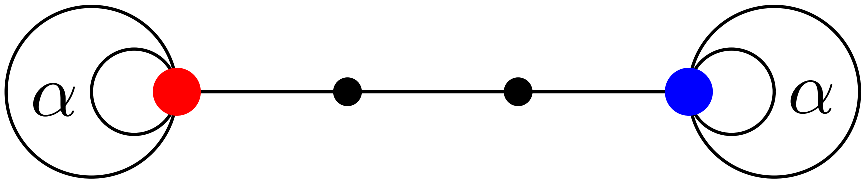


Conjecture (Casaccino, Lloyd, Mancini, and Severini '09)

For any n, one can find α so that there is perfect state transfer from ● to ● in P_n.

Perfect state transfer on paths revisited



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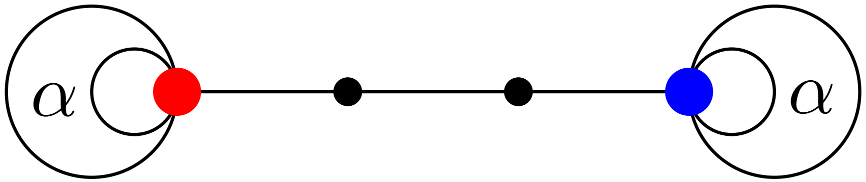

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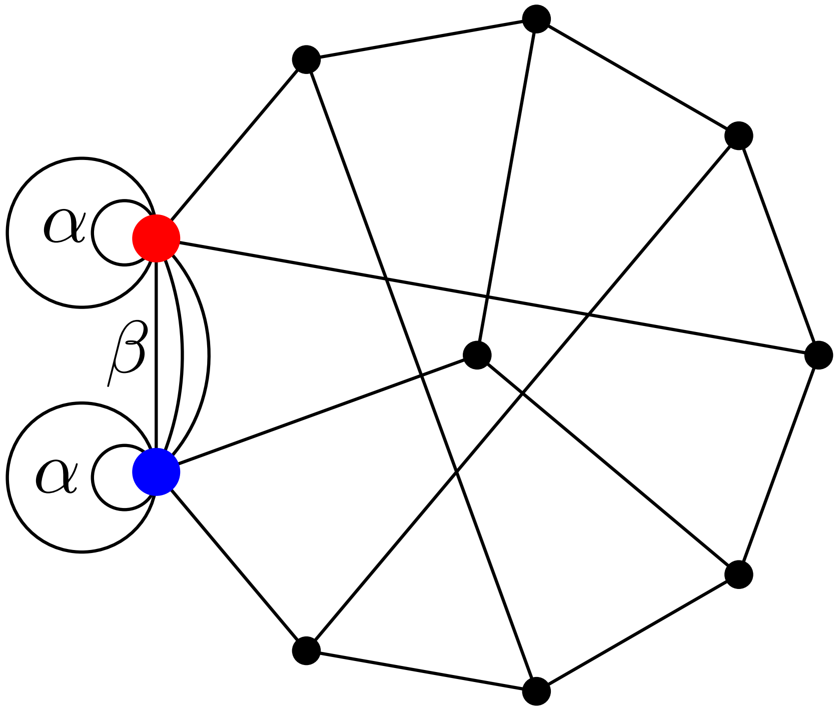
Perfect state transfer in strongly regular graphs



(α, β)-perturbation

Theorem (Godsil, Guo, Kempton and Lippner 2019)

For any strongly regular graph coming from an orthogonal array, there exists α and β such that the (α, β)-perturbation admits perfect state transfer.

UvA

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In fact, the "good" values of α, β are dense in the reals.

Discrete-time Quantum Walks

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UvA

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$$U = (2P - I)(2Q - I).$$

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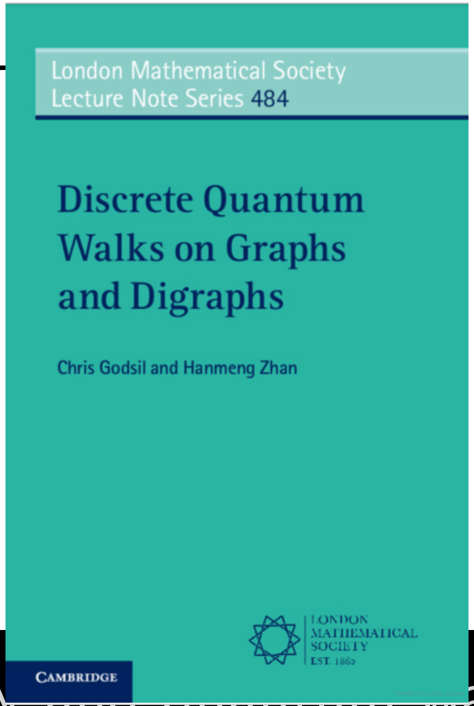
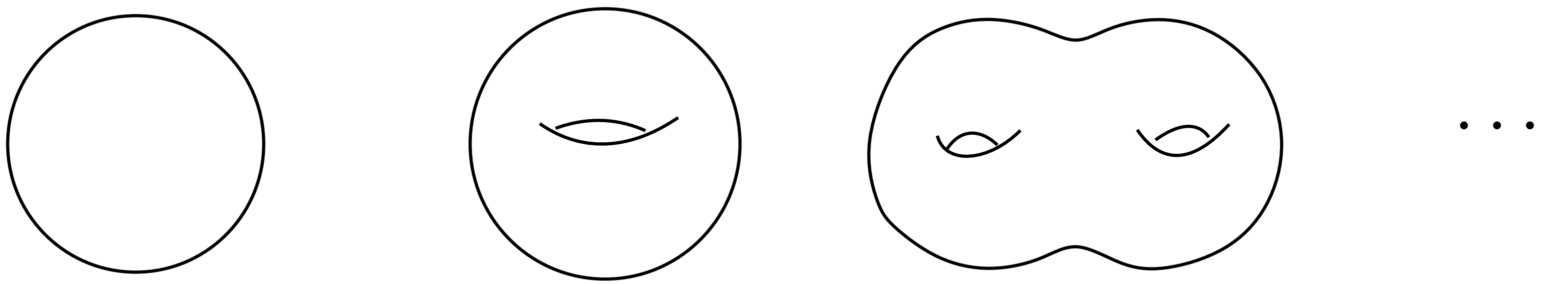


Graph embeddings



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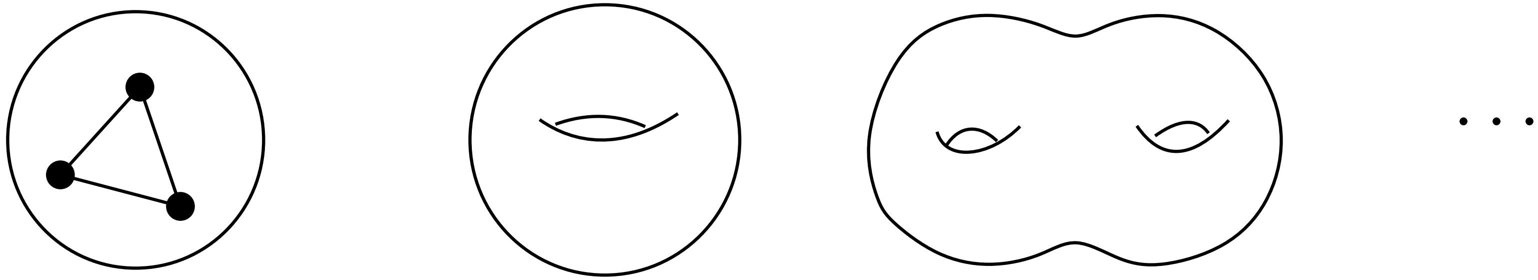

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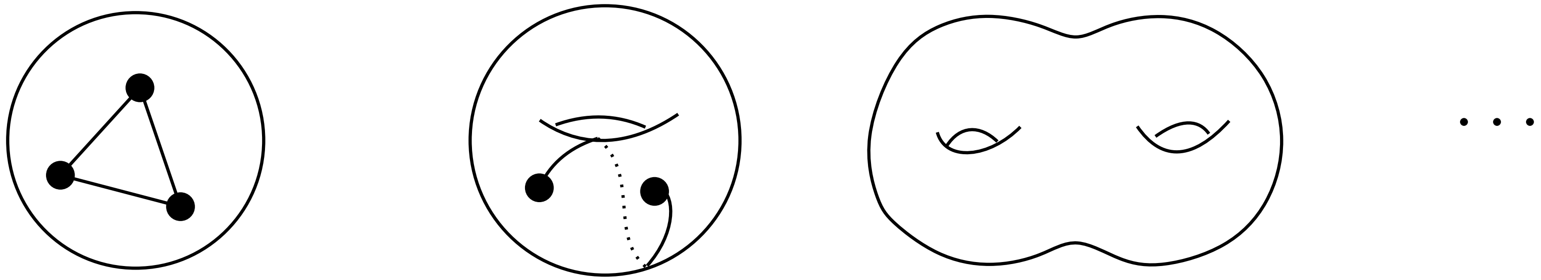

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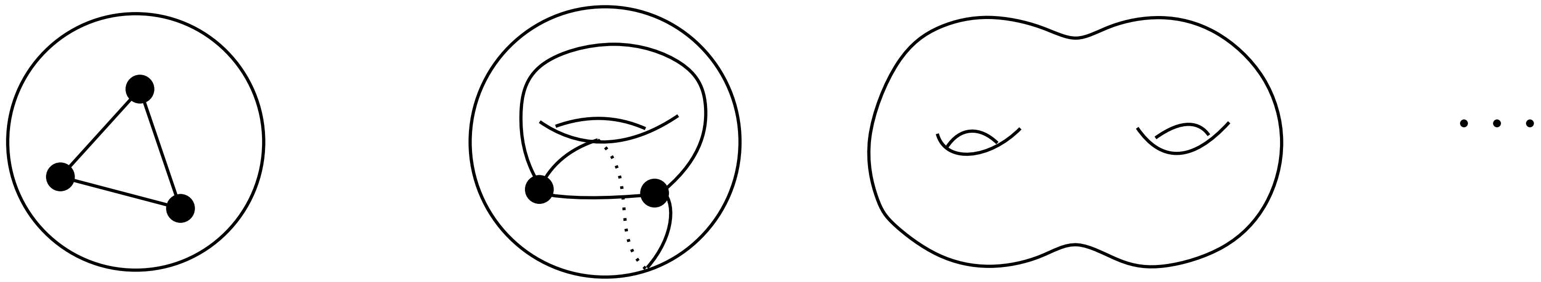

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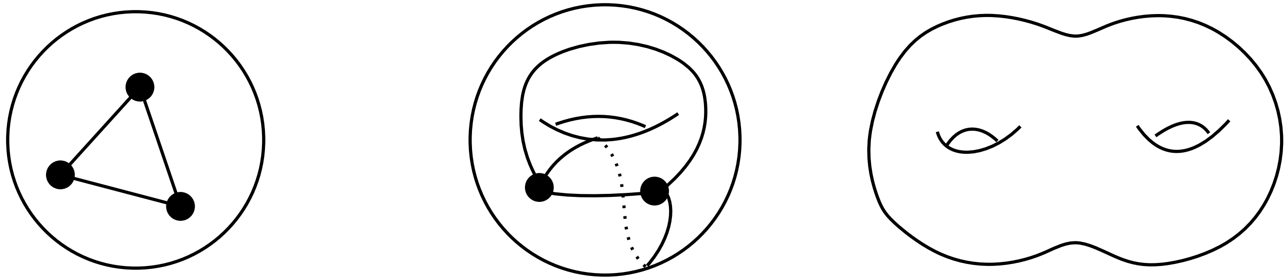
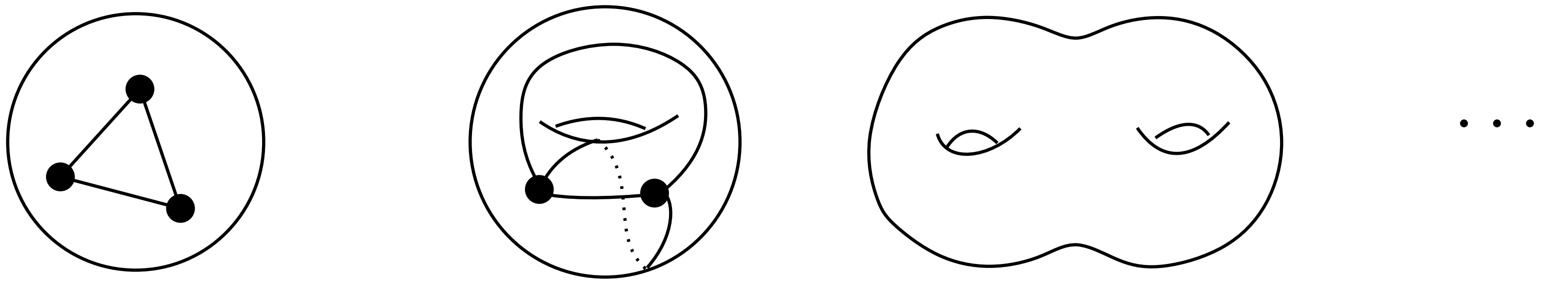


$\cdots$

Intuitively, we draw graphs on (orientable) surfaces such that the edges do not cross and "uses" the handles.

Graph embeddings



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This divides the surface in to regions called *faces*, such that each edge is on two faces.

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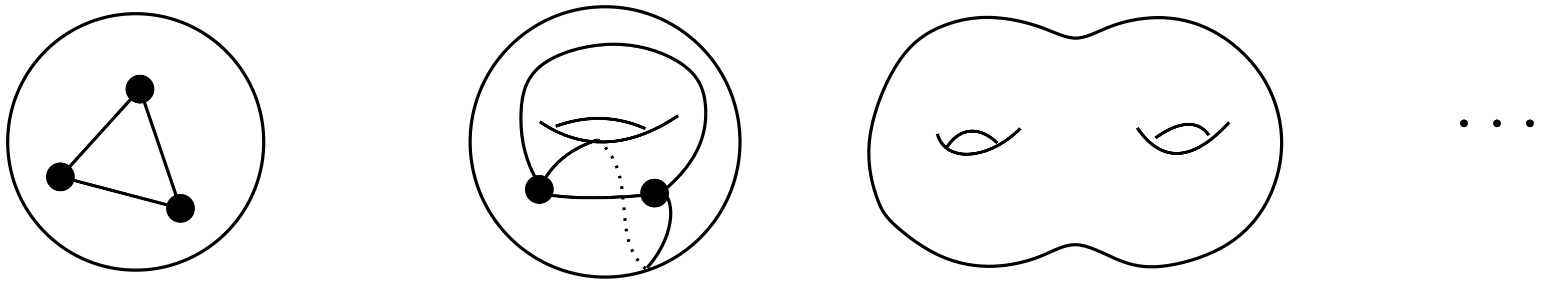

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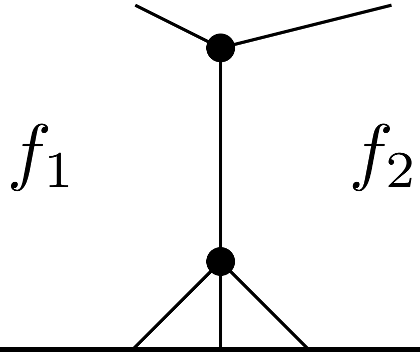

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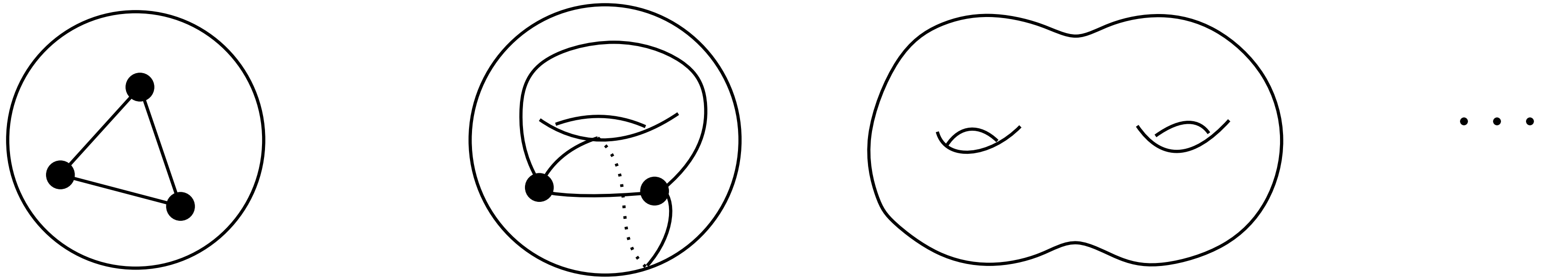

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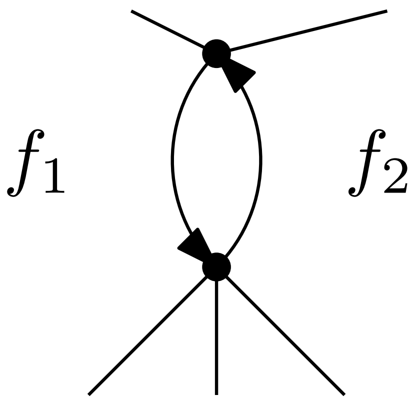

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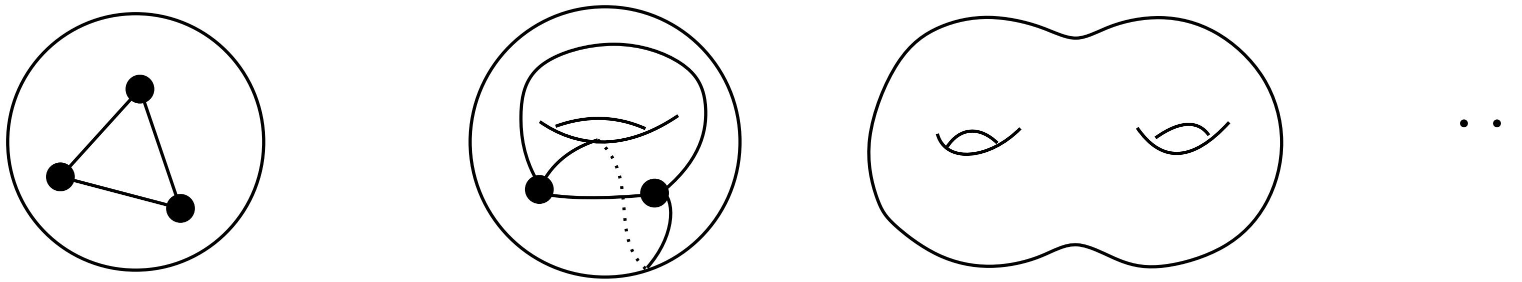

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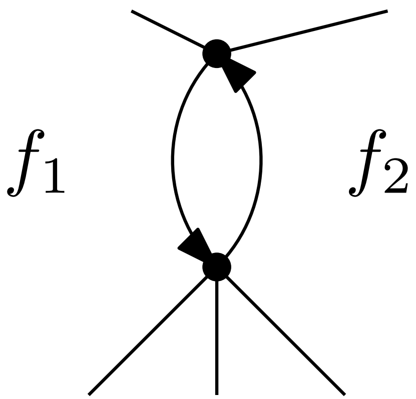
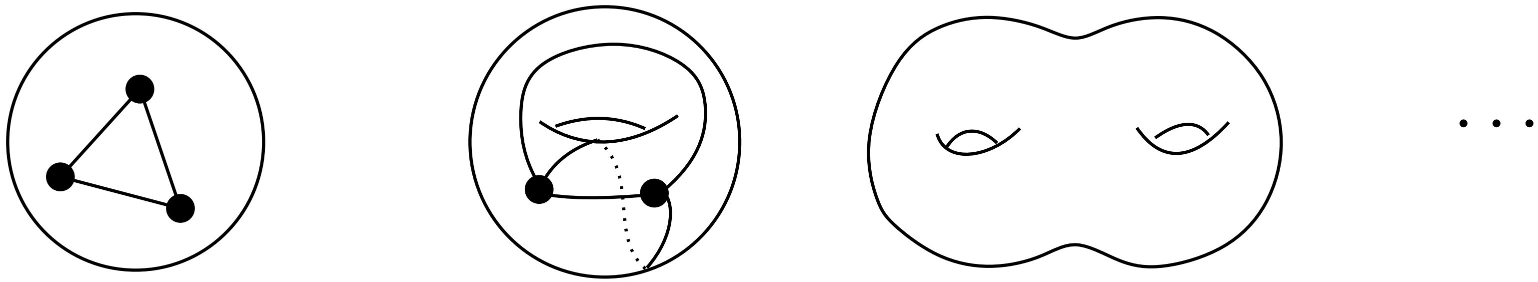


M is the arc-face incidence matrix and we take $P = MM^T$.

Graph embeddings



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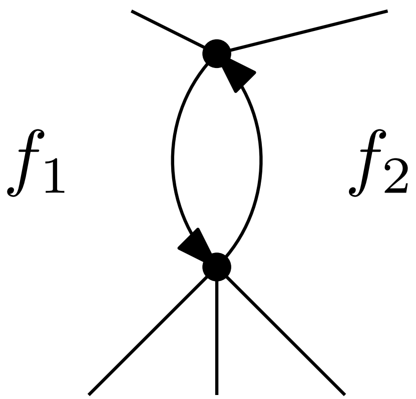

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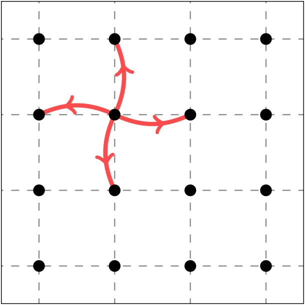
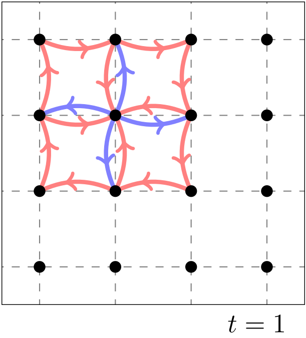
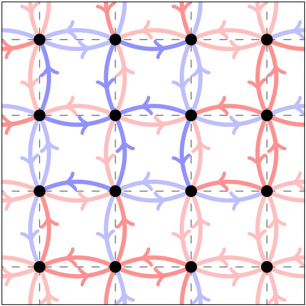
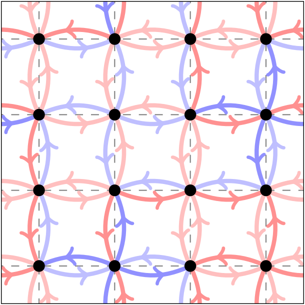
First considered by Zhan in 2020, generalizing various walks on the toroidal grid.



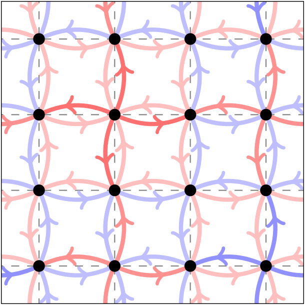
$t = 0$


$t = 1$

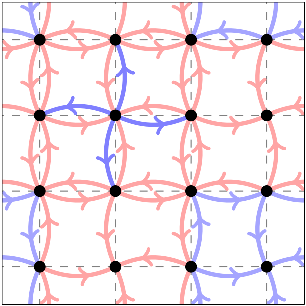

$t = 2$

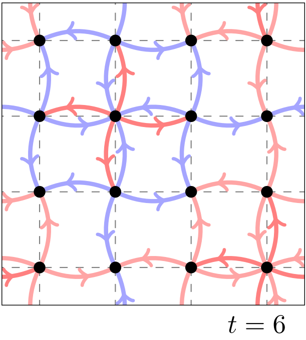


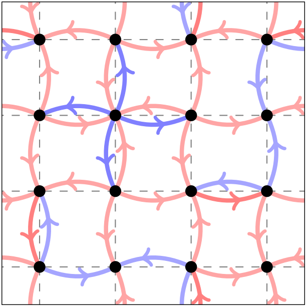
$t = 3$



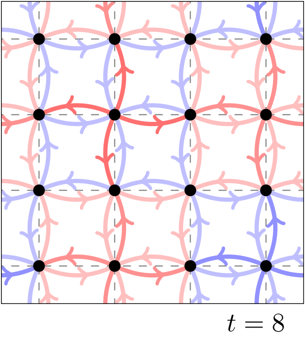
$t = 4$


$t = 5$

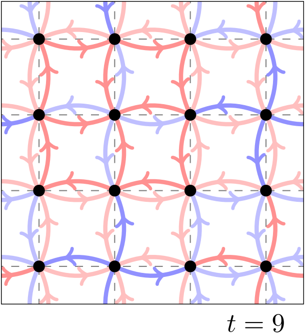

$t = 6$

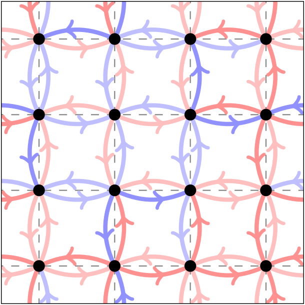


$t = 7$

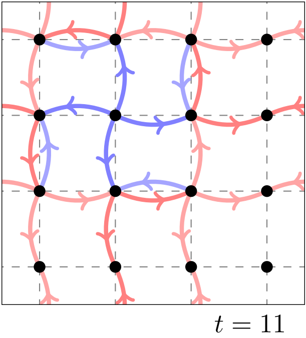


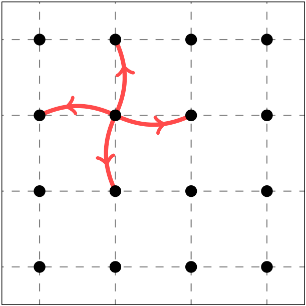
$t = 8$

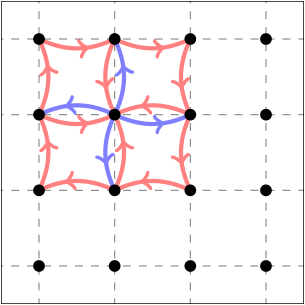

$t = 9$



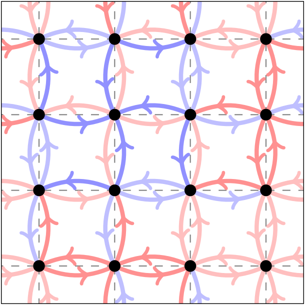
$t = 10$


$t = 11$


$t = 12$



$t = 13$



$t = 14$

State transfer

Perfect state transfer

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Define $B_t = N^T U^T N$.

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UvA

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Theorem (Guo & Schmeits 2022+)

For any two reflection walk, $B_t = T_t(B_1)$, where T_t is the tth Chebyshev polynomial of the first kind.

Computations

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Only the values $s = 1, 2, 6, 12$ appeared in these computations.

Conjecture

Let X be an orientably-regular map, and let U be its transition matrix. If $s > 0$ is such that $U^s = I$ and $U^r \neq I$ for all $r < s$, then $s \in \{1, 2, 6, 12\}$.

UvA

Lemma (Guo & Schmeits 2022+)

Let X be a map for which an associated matrix has rational eigenvalues. Assume that $U^\tau = I$ for some $\tau > 1$ and $U^s \neq I$ for all $s < \tau$, then $\tau \in \{2, 3, 4, 6, 12\}$.

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Open problem

What are some topological properties (genus, etc.). which affect the quantum walk?

Extensions of cospectrality

Cospectral things

Cospectral graphs: graphs cospectral with respect to the adjacency matrix

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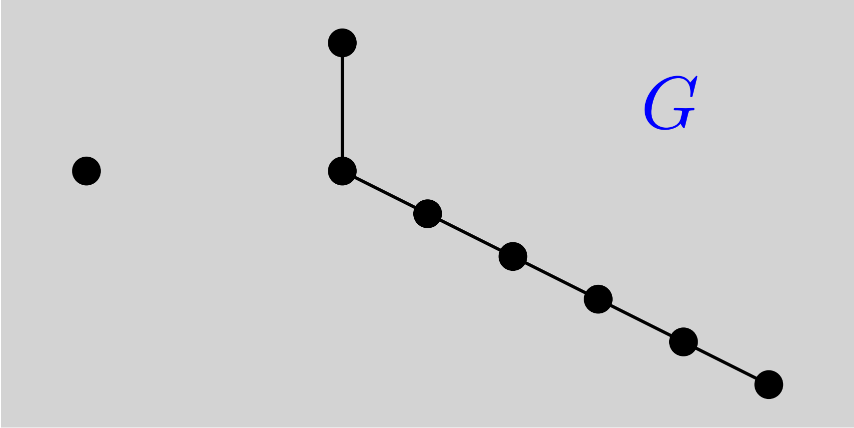

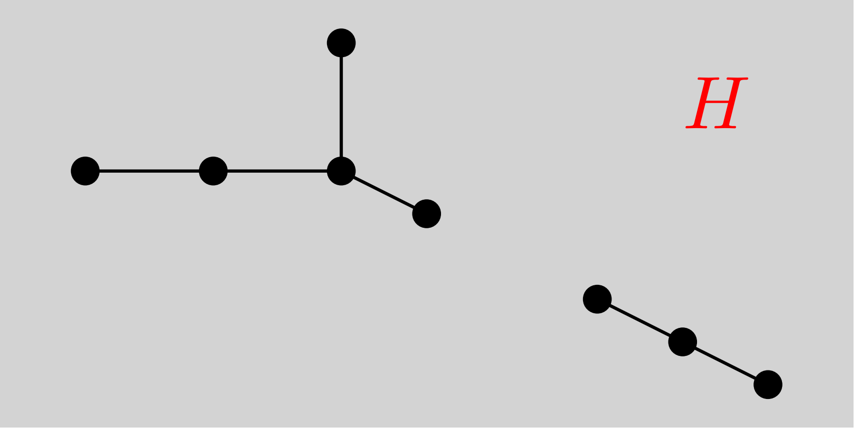

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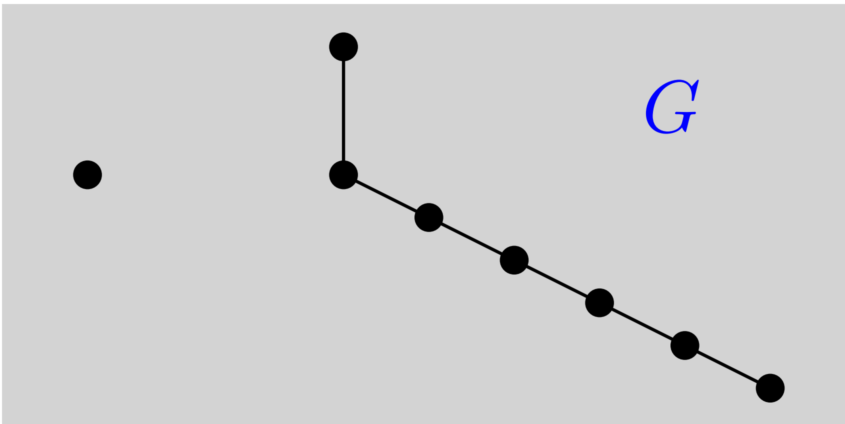

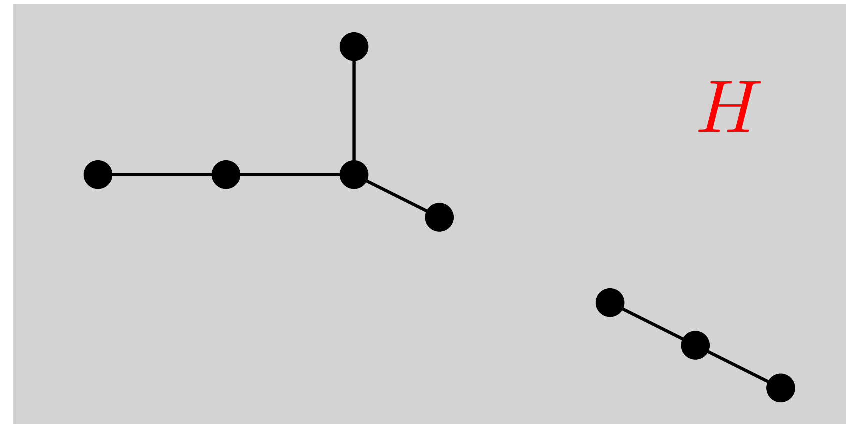

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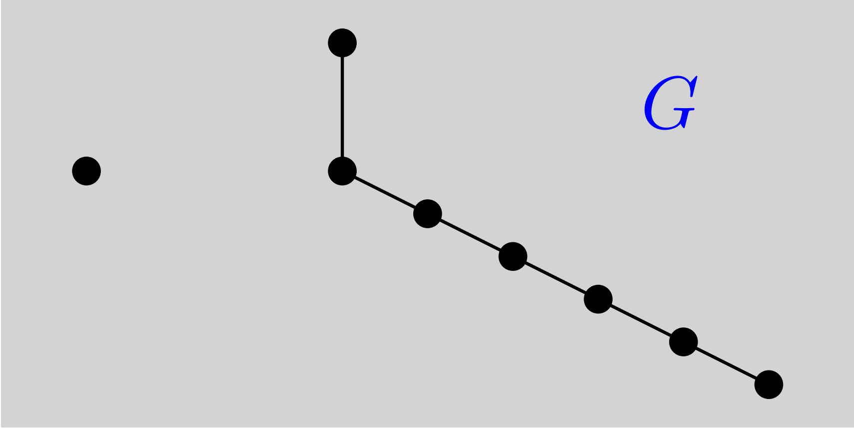

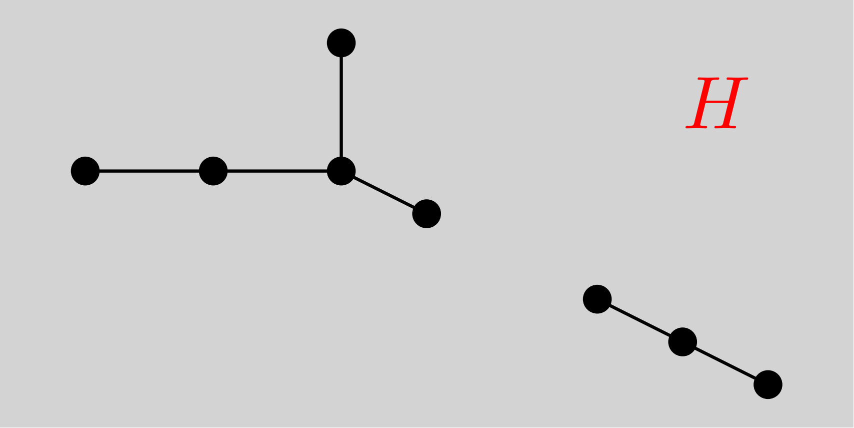

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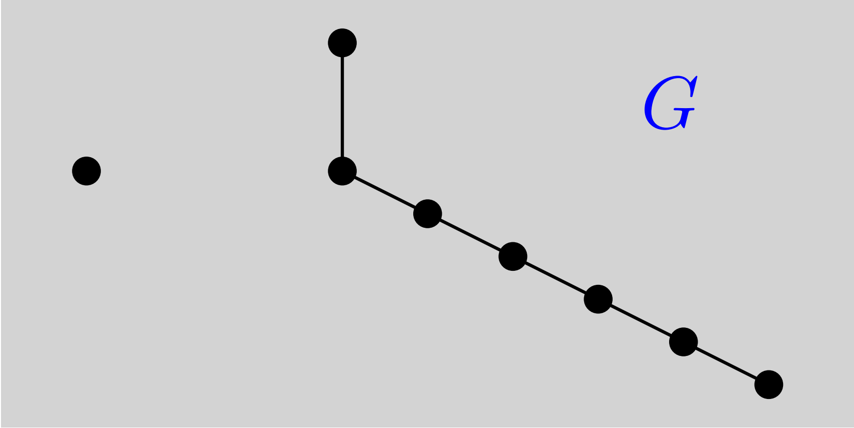

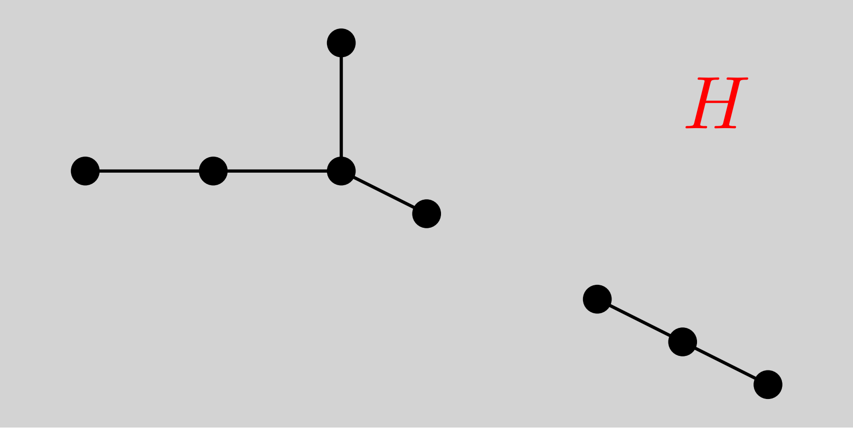

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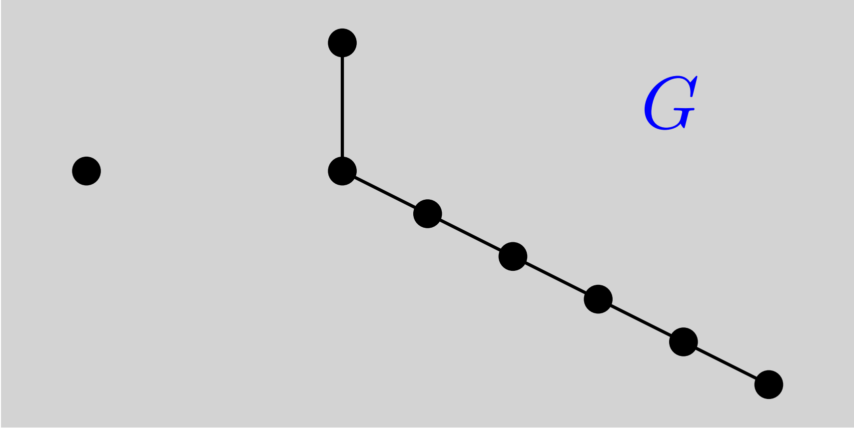

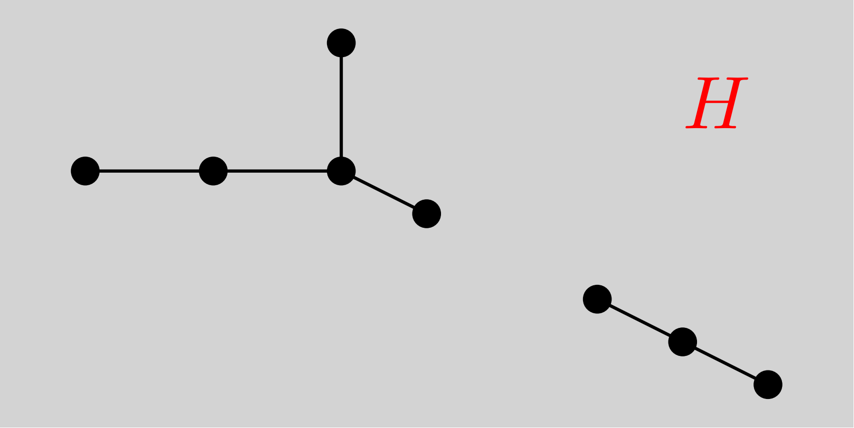

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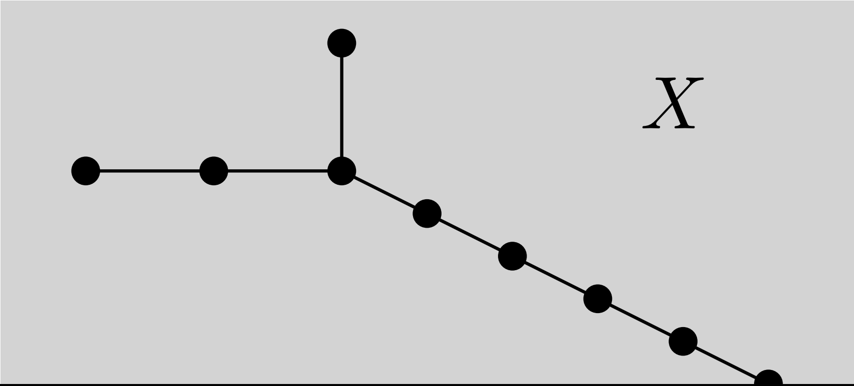

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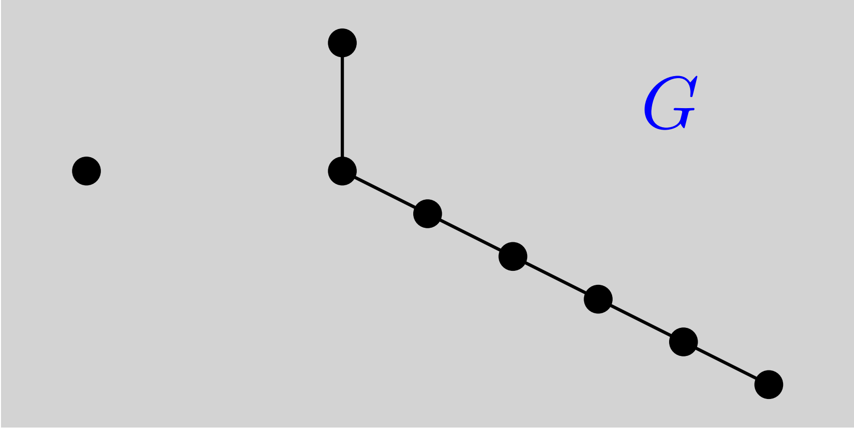

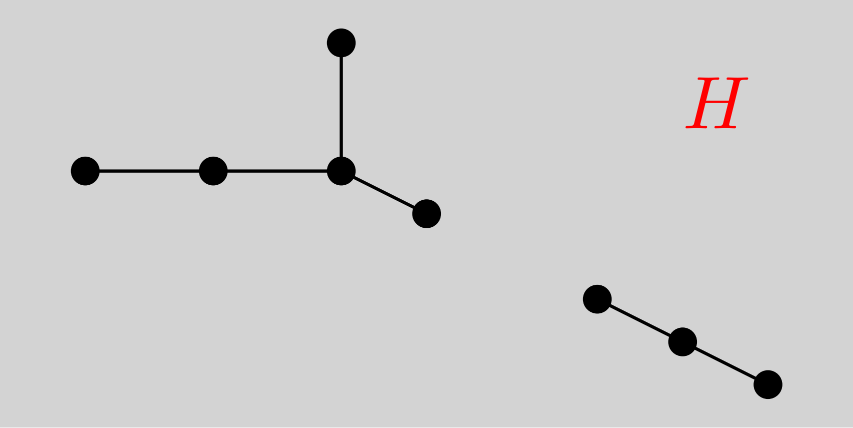

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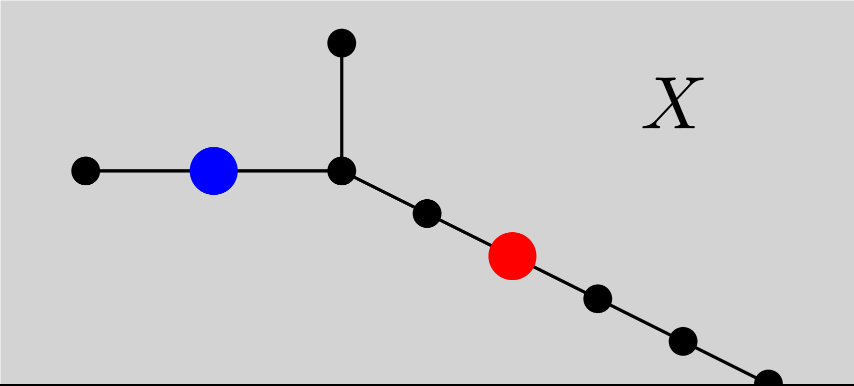

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But this is also given by the (y, y) entry of $A(Y)^k$.

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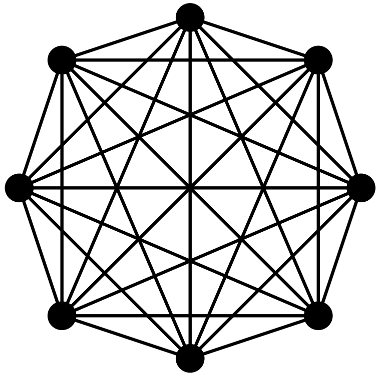
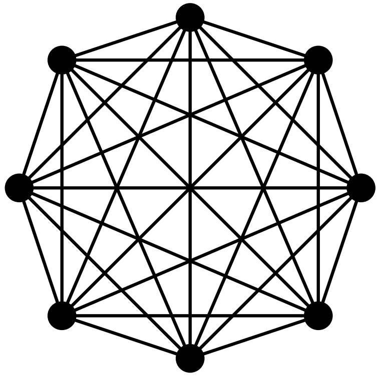
Non-examples



Complete graph K_n

Non-examples



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Every pair of vertices is cospectral.

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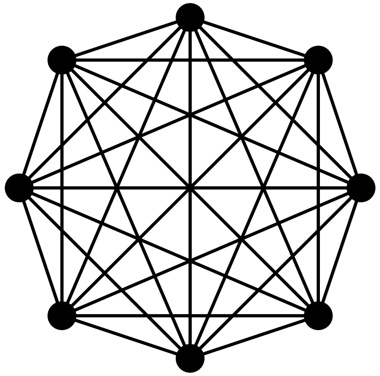

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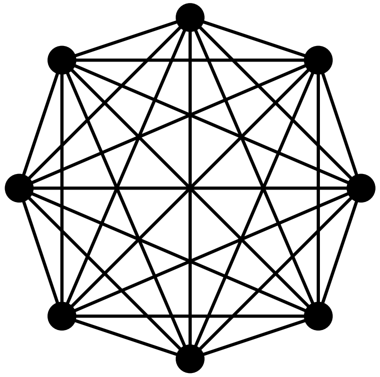

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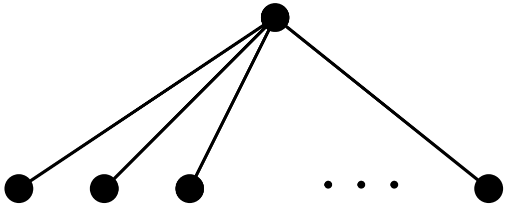

Star graph $K_{1,n}$

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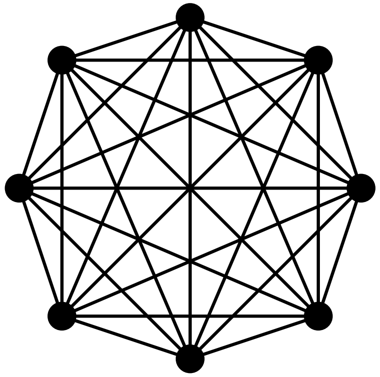

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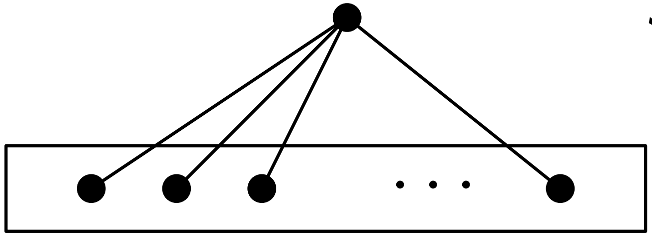

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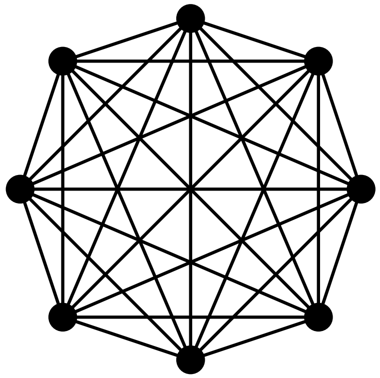

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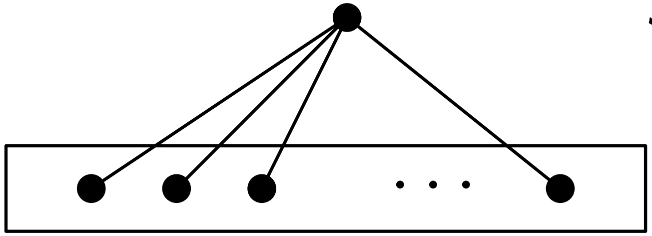

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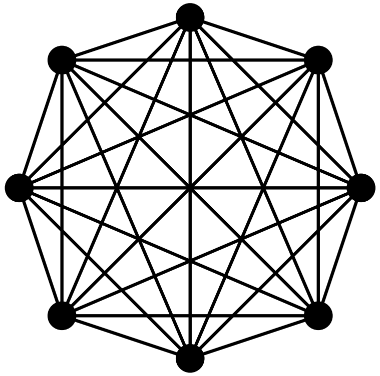

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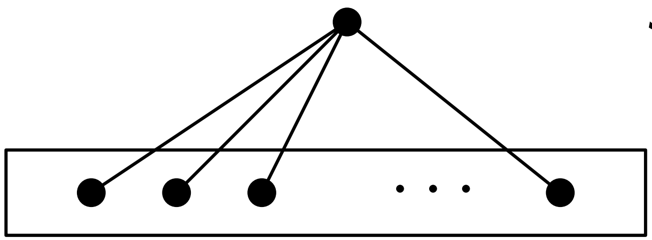

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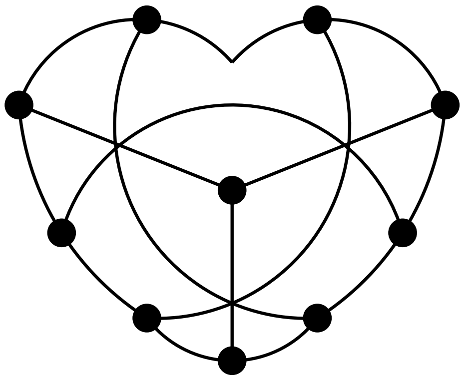

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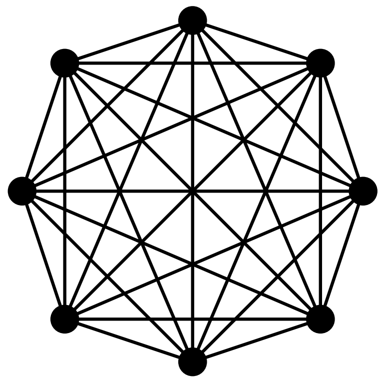

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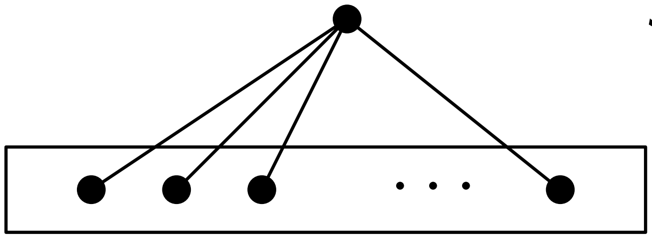

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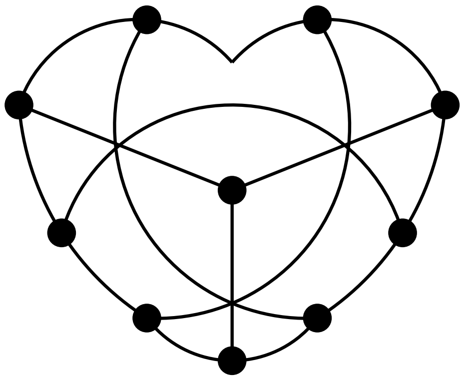
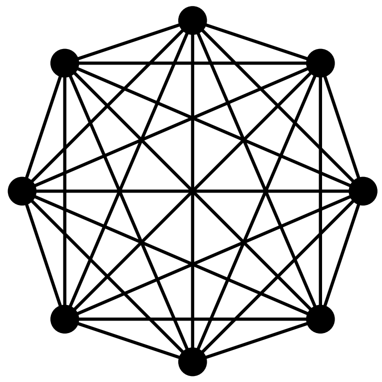


The Petersen graph

Non-examples



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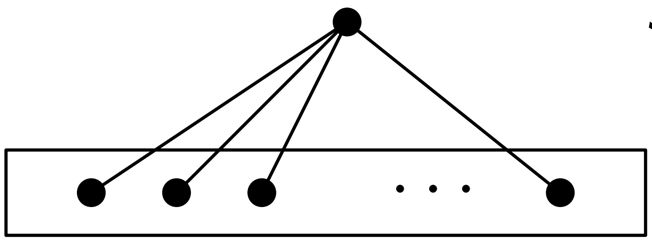

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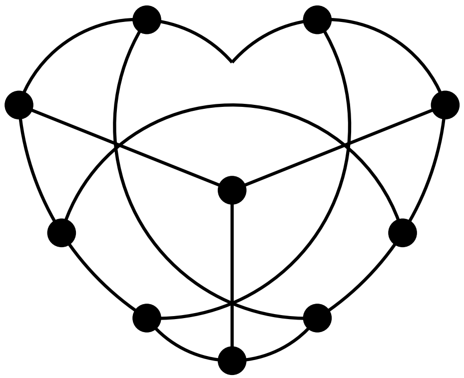

The Petersen graph

(Any primitive strongly regular graph)

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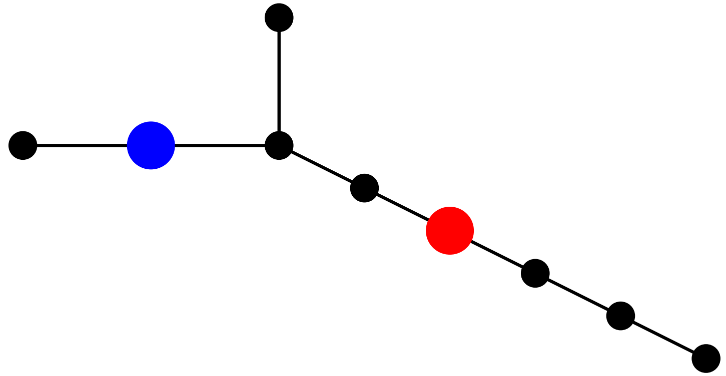
Examples



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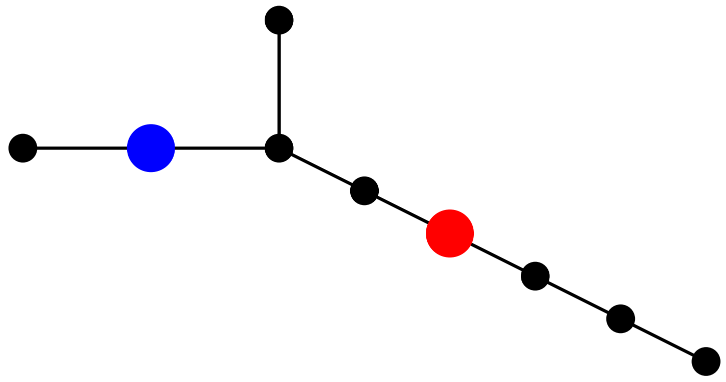

u, v cospectral and X has simple eigenvalues $\Rightarrow$ u, v strongly cospectral

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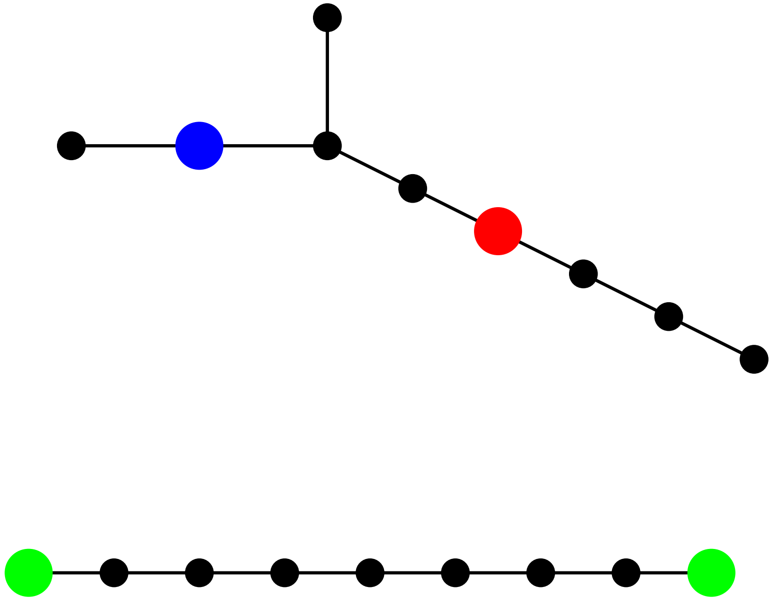

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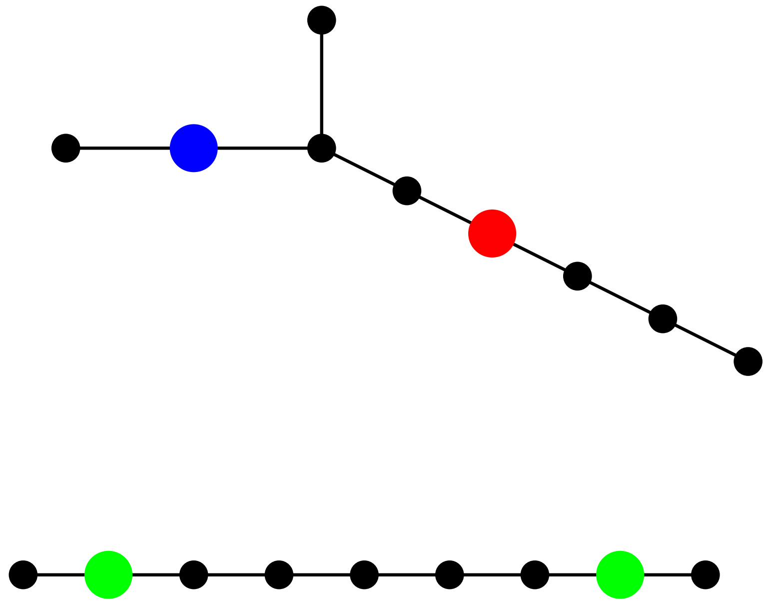

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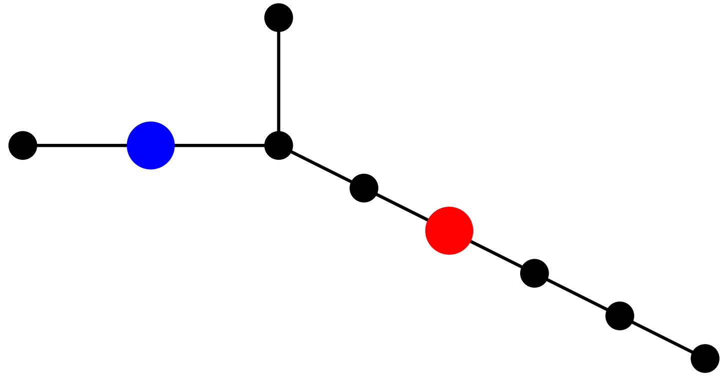

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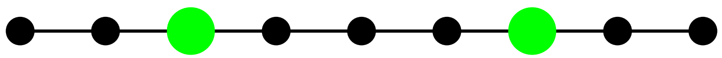

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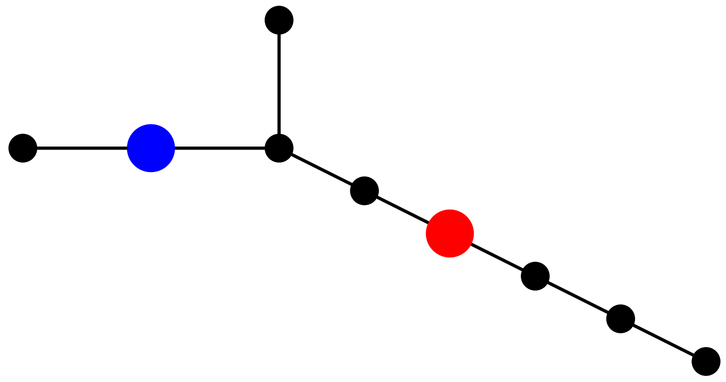
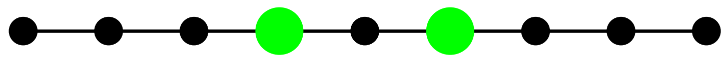


u, v cospectral and X has simple eigenvalues $\Rightarrow u, v$ strongly cospectral



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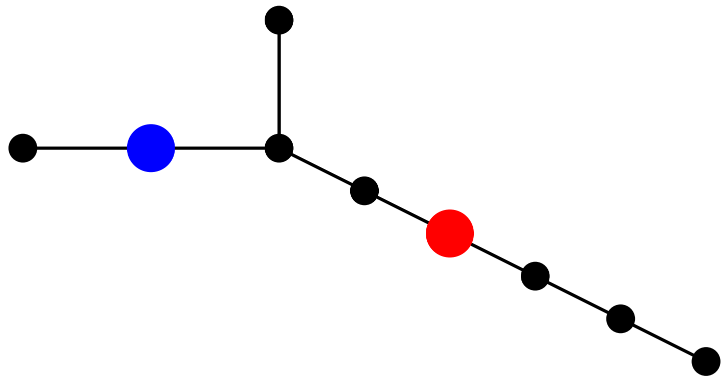

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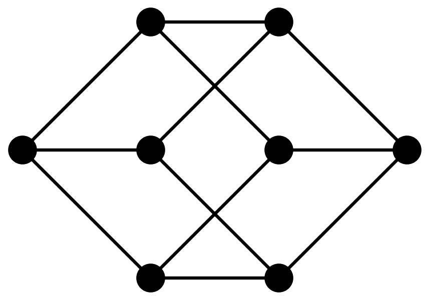

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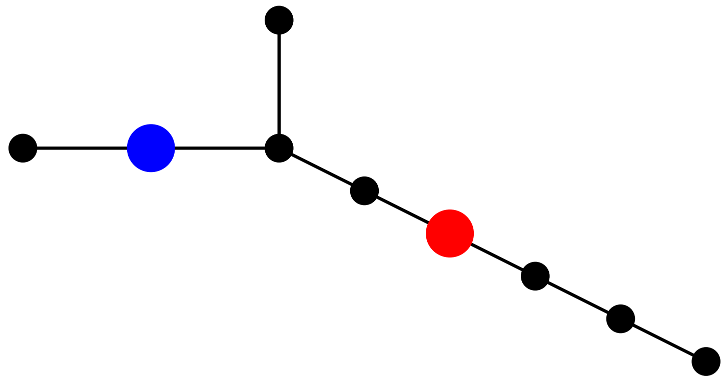

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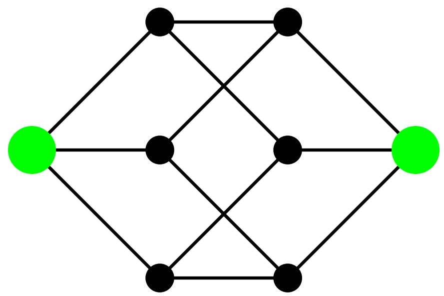
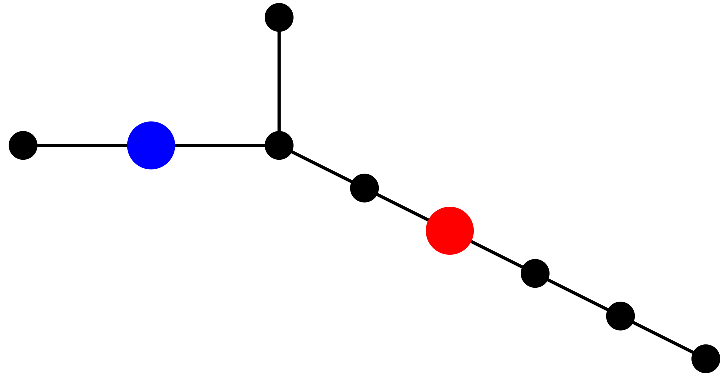


Antipodal vertices in the hypercube

Examples



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Theorem

Suppose B belongs to an association scheme. The following are equivalent.

(1) there exists x and y strongly cospectral mates w.r.t. B;

(2) there exists j such that A_j is a permutation matrix of order two with no fixed points; and

(3) every $x \in V$ has a strongly cospectral mate with respect to B.

Cospectral vertices

Let B be any matrix indexed by the vertices of the graph and let $B = \sum_\theta \theta E_\theta$.

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Theorem (Godsil 2012)

If the continuous-time quantum walk on G admits perfect state transfer from u to v then u, v are strongly cospectral.

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UvA

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Theorem (Guo & Schmeits 2022+)

If the vertex face quantum walk on G admits perfect state transfer from u to v at time τ then u, v are strongly cospectral with respect B_d for all d divisors of τ. In particular, they are strongly cospectral w.r.t. B_1.

(Orthogonal) Symmetries of Graphs

Orthogonal symmetries

Let X be a graph with adjacency matrix $A = \sum_{r=0}^{d} \theta_r E_r$.

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In all the examples listed so far, the strongly cospectral vertices come in pairs.

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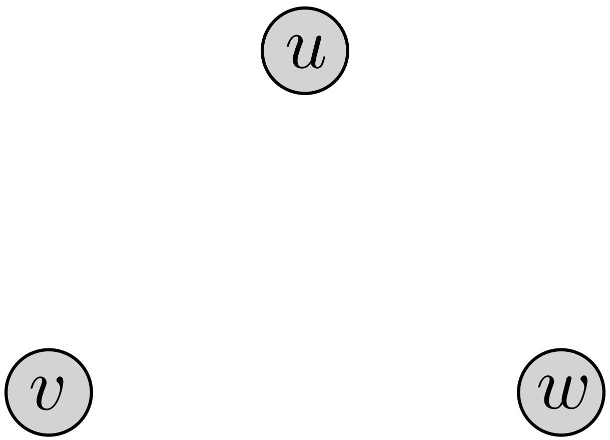

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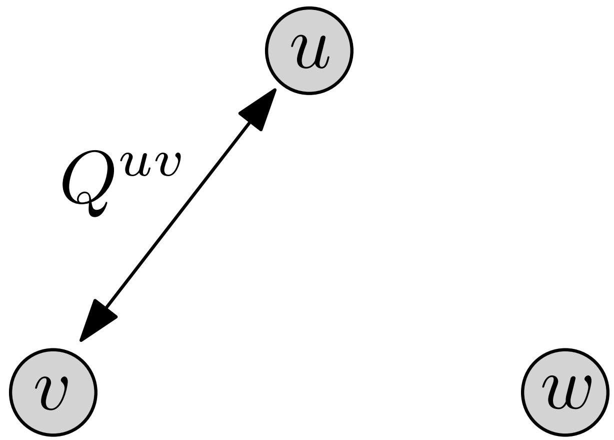

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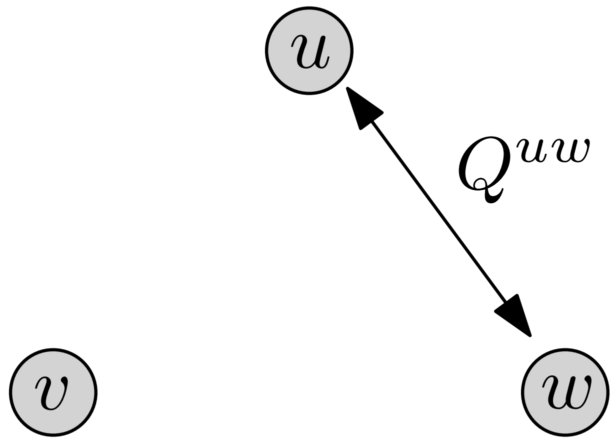

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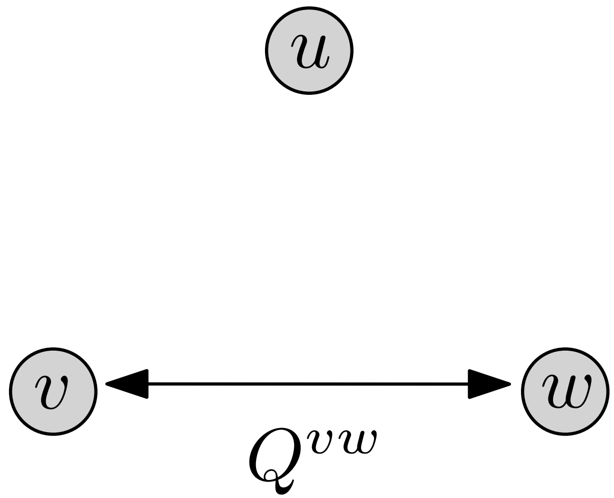

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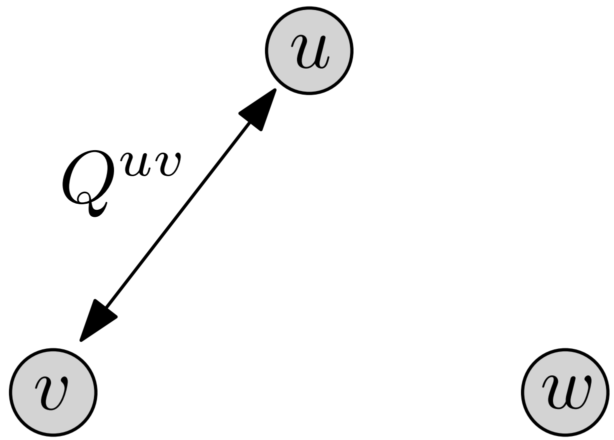

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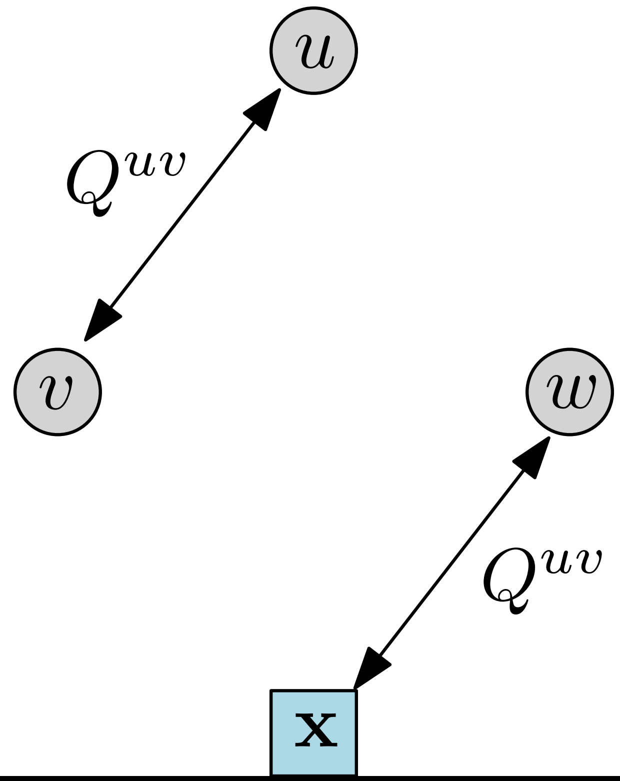

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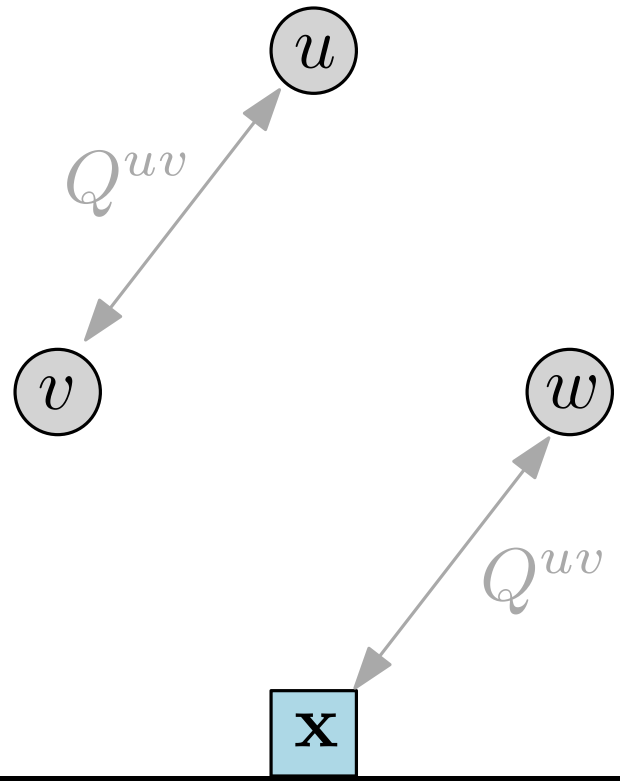

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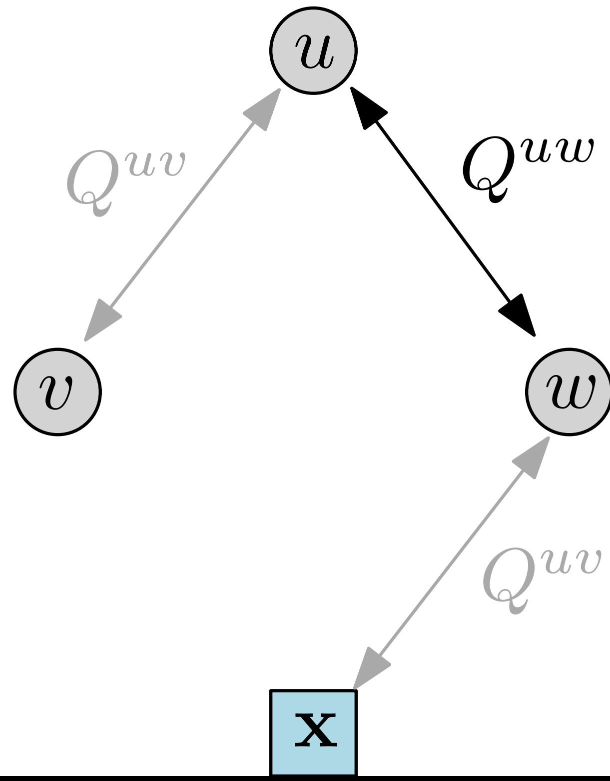

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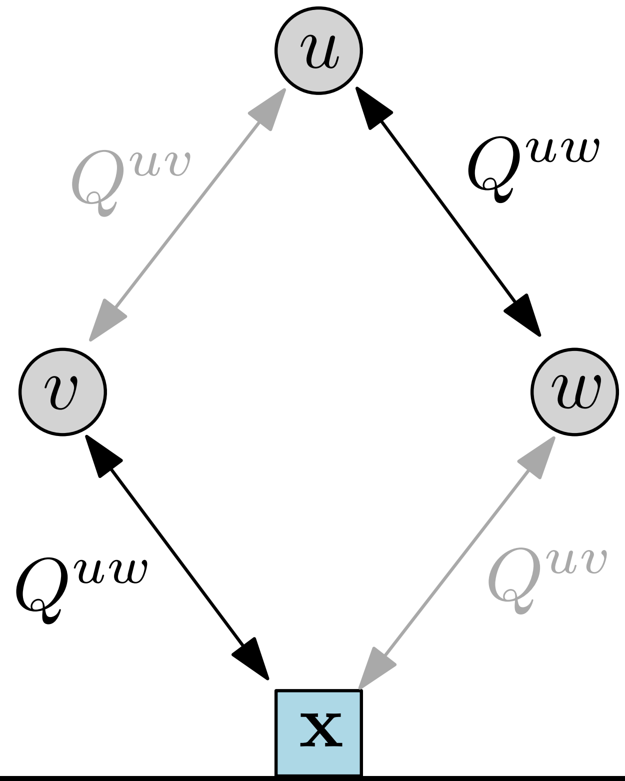

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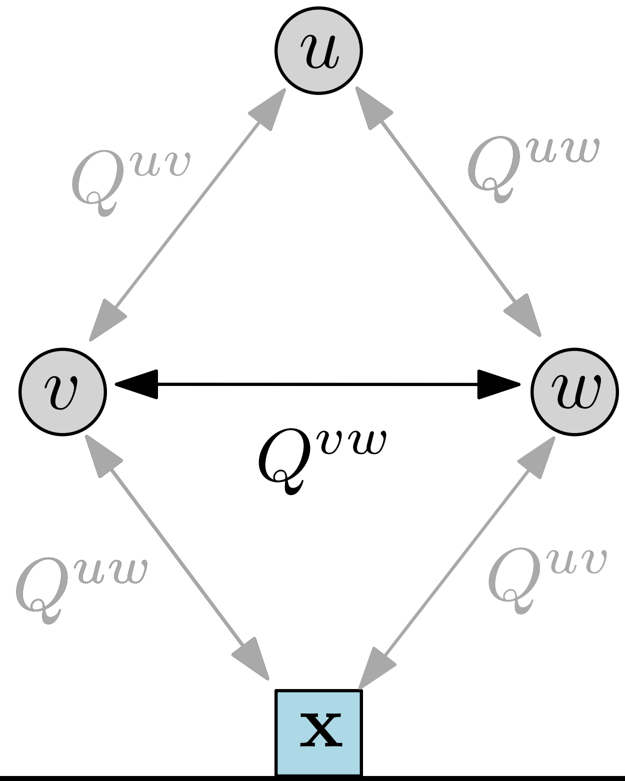

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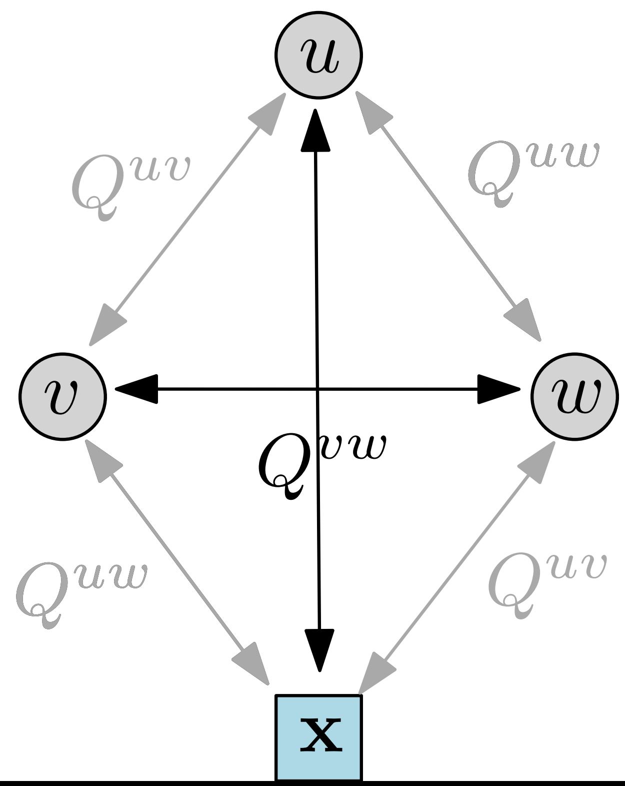

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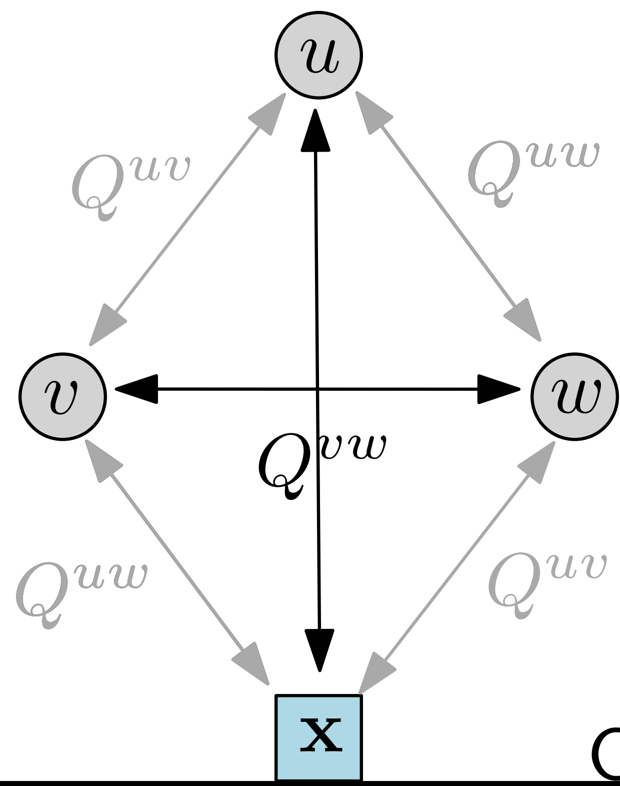

Chan, KG, Zhan 2023++

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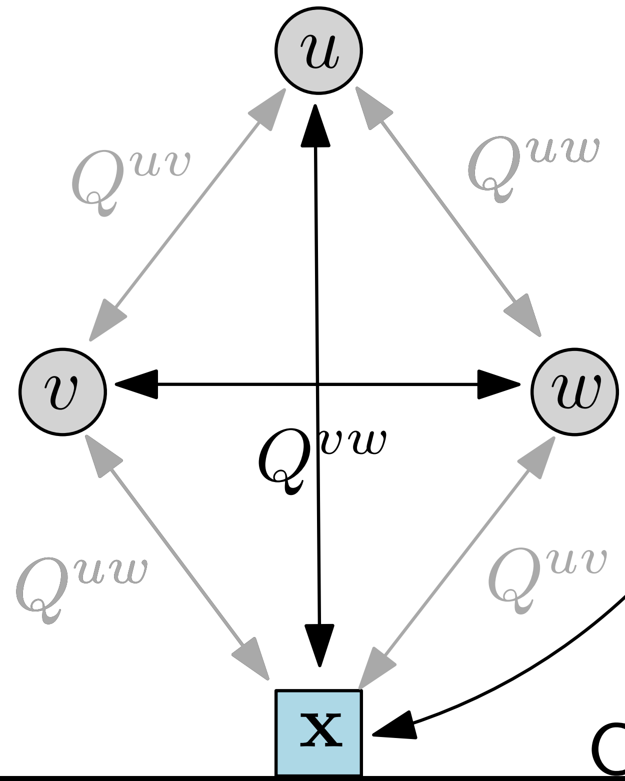

This vector $\mathbf{x}$ is an example of a phantom mate.

Chan, KG, Zhan 2023++

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- One can study quantum walks using linear algebraic graph theory and prove properties about the walk using algebraic properties of the graph.
- In the process of doing this, various new (completely classical) graph properties arise and provide interesting combinatorial problems.

Thanks!