Powers of planar graphs, product structure, and blocking partitions

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Blocking Partitions

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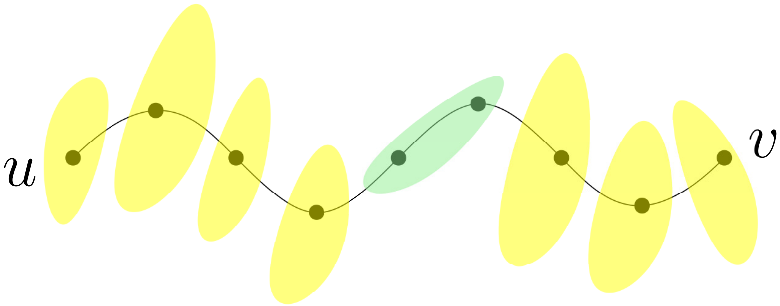

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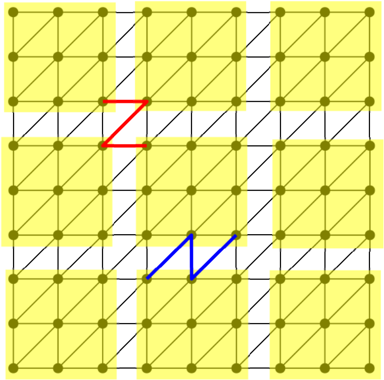

3-blocking partition of width 9

Blocking Partitions and Maximum Degree

Question: For a graph class $\mathcal{G}$, does there exists an $\ell \in \mathbb{N}$ and a function f such that every graph $G \in \mathcal{G}$ has an ℓ-blocking partition of width at most $f(\Delta(G))$?

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Proposition There are no constants $\ell, w \in \mathbb{N}$ such that every 4-regular graph G has an ℓ-blocking partition of width at most w

Planar Graphs

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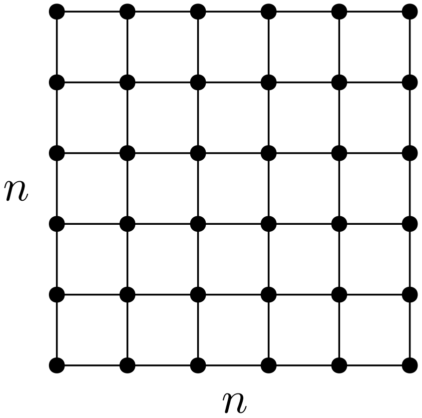

Main Result

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Theorem [Distel, H., Seweryn, Wood '23]

There is a function f such that every planar graph G with maximum degree Δ has a 222-blocking partition with width at most $f(\Delta)$

Planar Graph Product Structure Theorem

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Theorem [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '20]
For every planar G, there is a graph H of treewidth at most 8 and a path P such that G is a subgraph of $H \boxtimes P$

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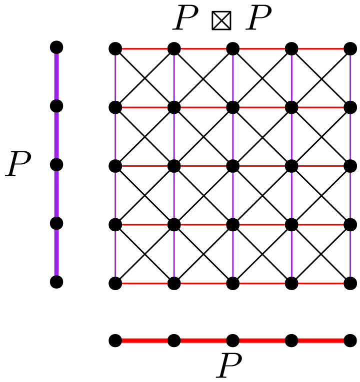

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Note that $\mathrm{tw}(H) \leqslant (3+1) * 3 - 1 = 11$

Applications of Graph Product Structure Theory

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Resolved important open problems

- queue-number [DJMMUW '20]
- nonrepetitive colourings [DEJWW '20]
- centred colourings [DFMS '20]
- labelling schemes [BGP '20]
- adjacency labelling schemes [DEGJMM '20]
- twin-width [BHLK '22, BKW '22]
- clustered colouring [DEMW '23]

Graph Product Structure Theory

Other graph classes with product structure includes

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- bounded pathwidth [DHJMMW '23]
- bounded treewidth [CCDGHHHITTW '23, ISW '22, DHHJLMMRW '24]
- bounded Euler genus [DJMMUW '20, DHHW '22]
- apex-minor-free graphs [DJMMUW '20]
- map graphs [DMW '23]
- fan-planar graphs [HW '21]
- fan-bundle planar graphs [HW '21]
- k-planar graphs [DMW '23]
- powers of planar graphs [DMW '23, HW '21]

Powers of Planar Graphs

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For a graph G, the k-th power G^k of G is the graph G^k where $V(G^k) := V(G)$ and $uv \in E(G^k)$ if and only if $\mathrm{dist}_G(u, v) \leqslant k$

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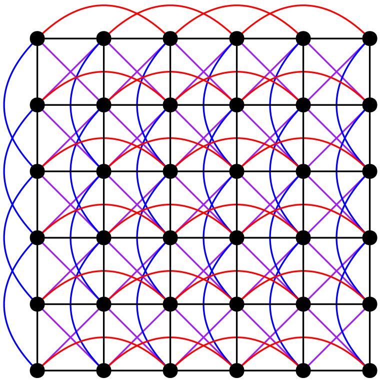

Square of the $n \times n$ grid

Product Structure for Powers of Planar Graphs

Theorem [Dujmović, Morin, Wood '23]

For every $k \in \mathbb{N}$, for every planar graph G with maximum degree Δ, G^k is contained in $H \boxtimes P \boxtimes K_{O(k^4 \Delta^k)}$, for some graph H of treewidth $O(k^3)$ and some path P

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Question [Ossona De Mendez '21]

Is there a function f and a universal constant C such that for every planar G with maximum degree Δ, G^k is contained in $H \boxtimes P \boxtimes K_{f(k,\Delta)}$ for some graph H with treewidth at most C?

Results

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Theorem [Distel, H., Seweryn, Wood '23]

There is a function f such that every planar graph G with maximum degree Δ, the graph G^k is contained in $H \boxtimes P \boxtimes K_{f(k,\Delta)}$ for some graph H with $\mathrm{tw}(H) \leqslant 963\,922\,179$

Proof Sketch: Blocking Partition

An ℓ-blocking partition of a graph G with width at most w is a partition of $V(G)$ into connected sets of size at most w such that every path of length greater than ℓ in G contains at least two vertices in one part

Theorem [Distel, H., Seweryn, Wood '23]

There is a function f such that every planar graph G with maximum degree Δ has a 222-blocking partition with width at most $f(\Delta)$

Proof Idea for Powers of Planar Graphs

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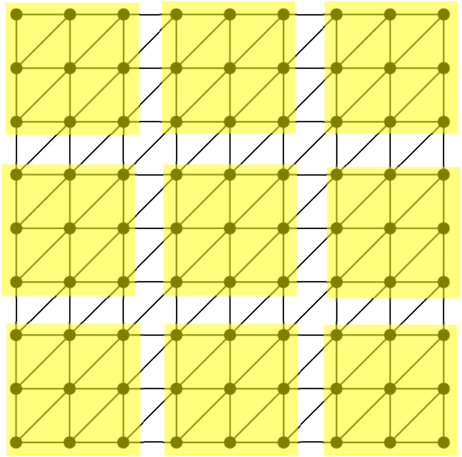
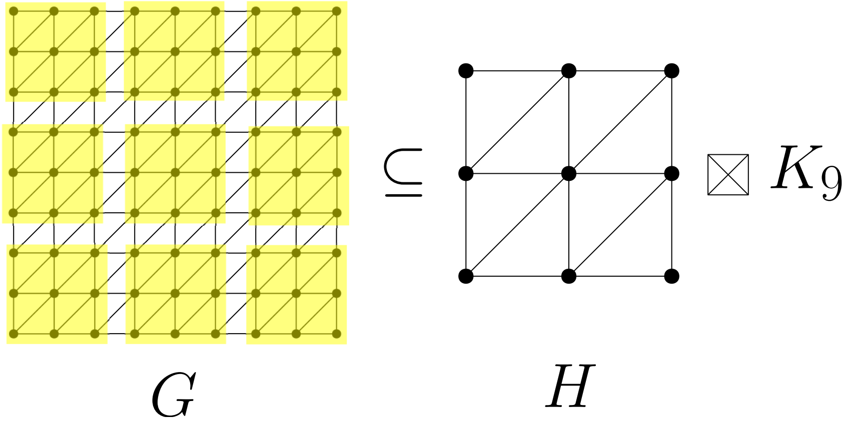


3-blocking partition of width 9

Proof for Graph Powers



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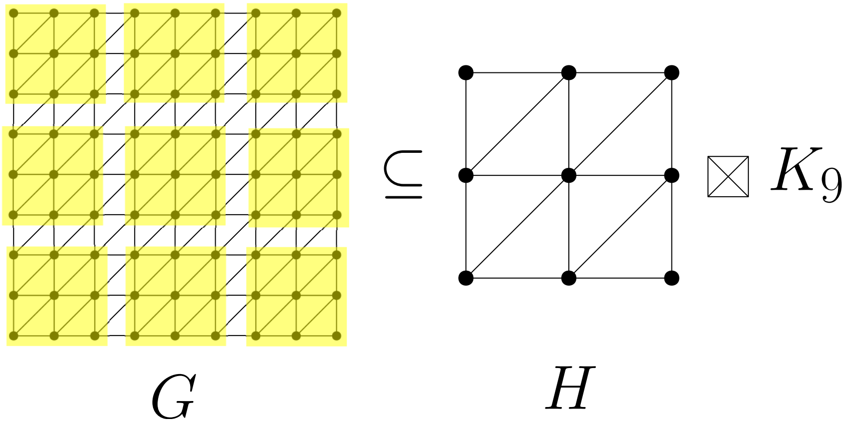
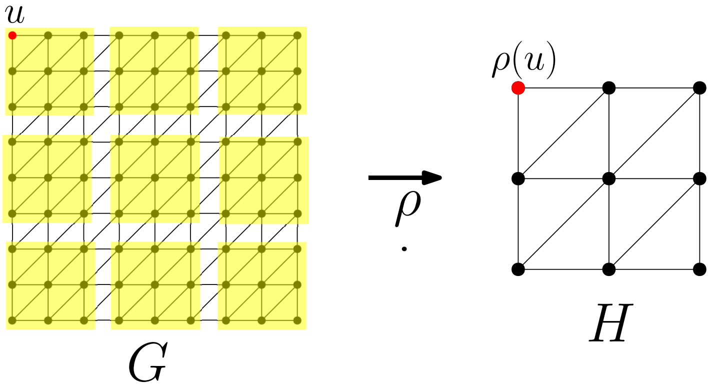


Observation H is a minor of G where $\Delta(H)$ is bounded by a function of $\Delta(G)$

Proof for Graph Powers

Let $\rho\colon V(G) \to V(H)$ be the map that corresponds to the blocking partition



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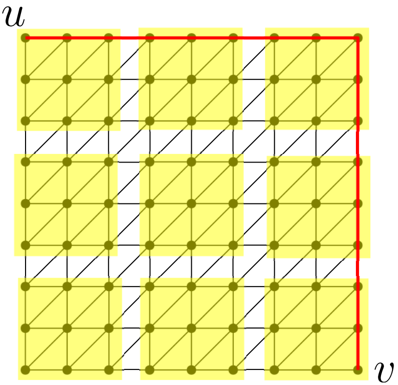

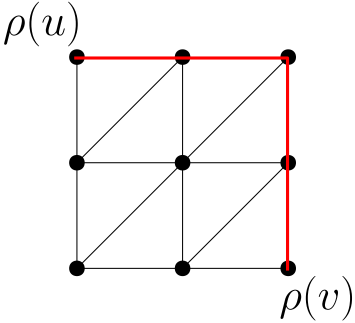

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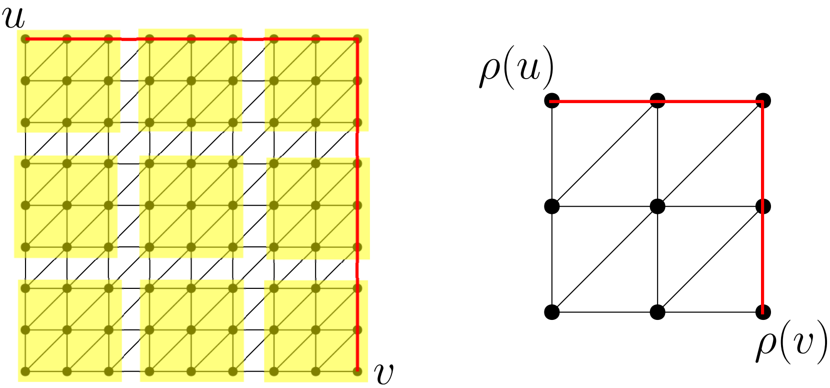

Observation For every $u, v \in V(G)$, if $\mathrm{dist}_G(u, v) \geqslant 223$ then

$$\mathrm{dist}_H(\rho(u), \rho(v)) < \mathrm{dist}_G(u, v)$$

Proof for Graph Powers

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Lemma [Distel, H., Seweryn, Wood '23]

For $\ell = 222$, there exists functions f and g, such that, for every $k \in \mathbb{N}$ and planar graph G,

$$G^k \subseteq H^\ell \boxtimes K_{f((\Delta(G),k))}$$

for some planar graph H with $\Delta(H) \leqslant g(\Delta(G), k)$

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Open Problems

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Does every proper minor-closed class admit blocking partition?

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Are there other applications of blocking partitions?