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Safe Sets and Dominating Sets of Graphs

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Joint work with
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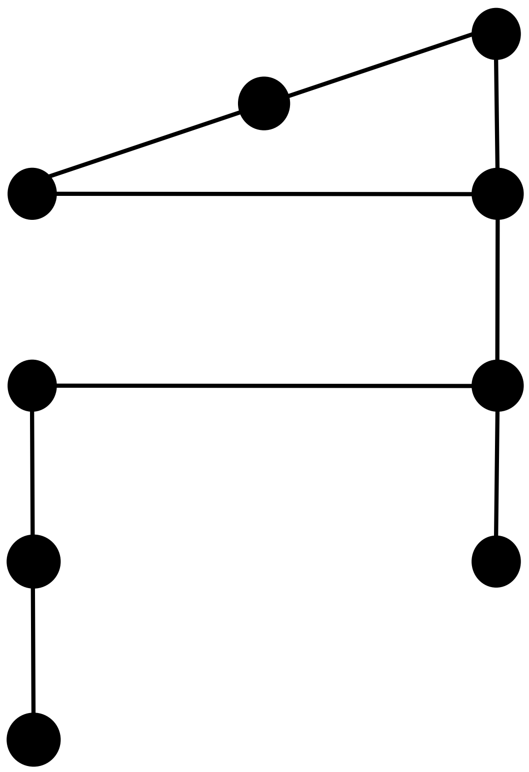
For a graph G, a vertex subset D is a dominating set if every vertex of G is either in D or adjacent to a vertex in D.



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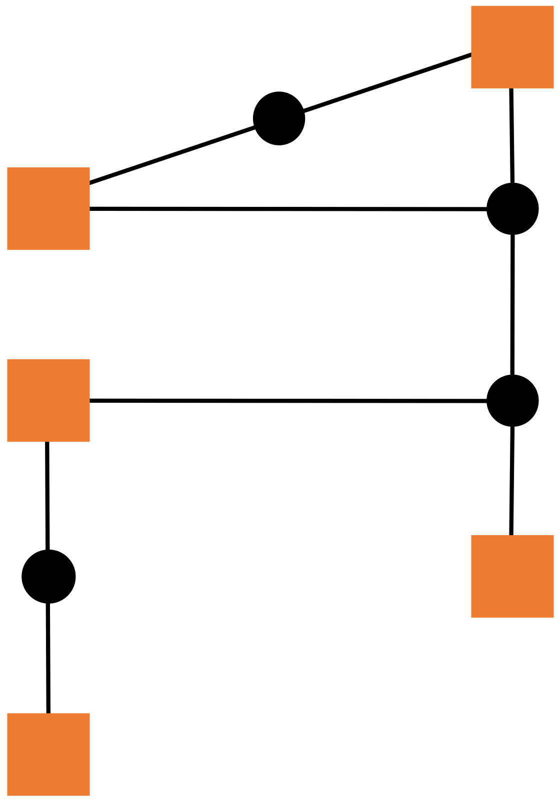

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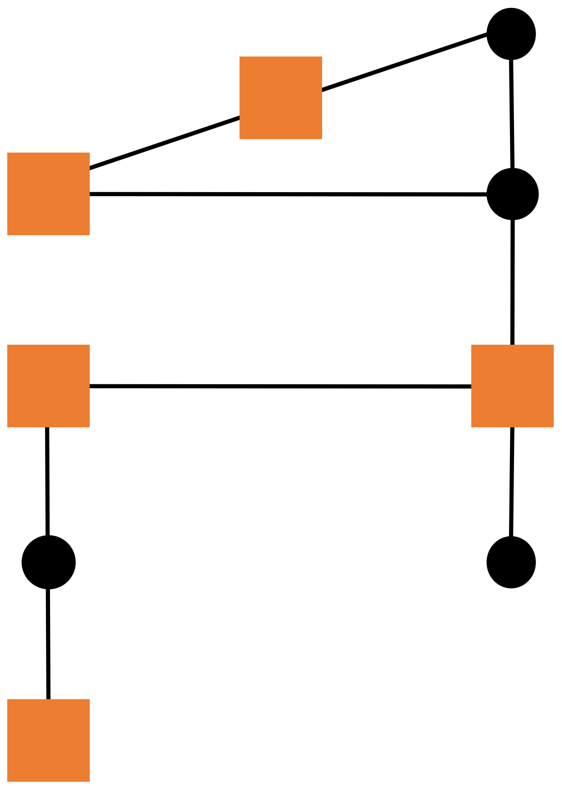

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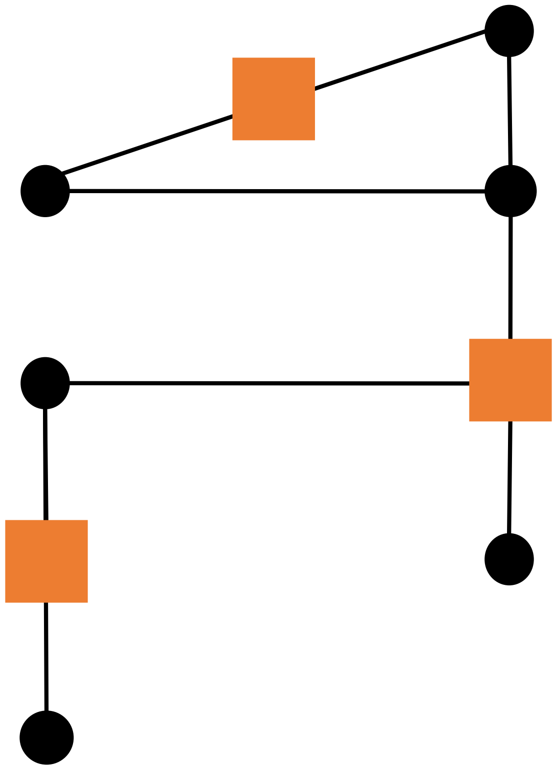
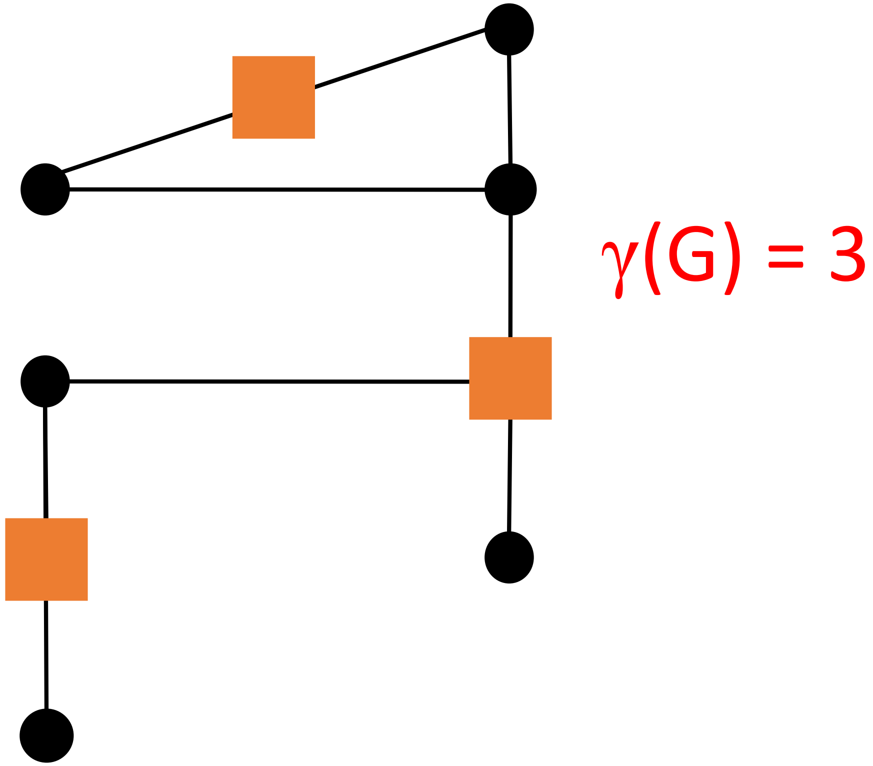
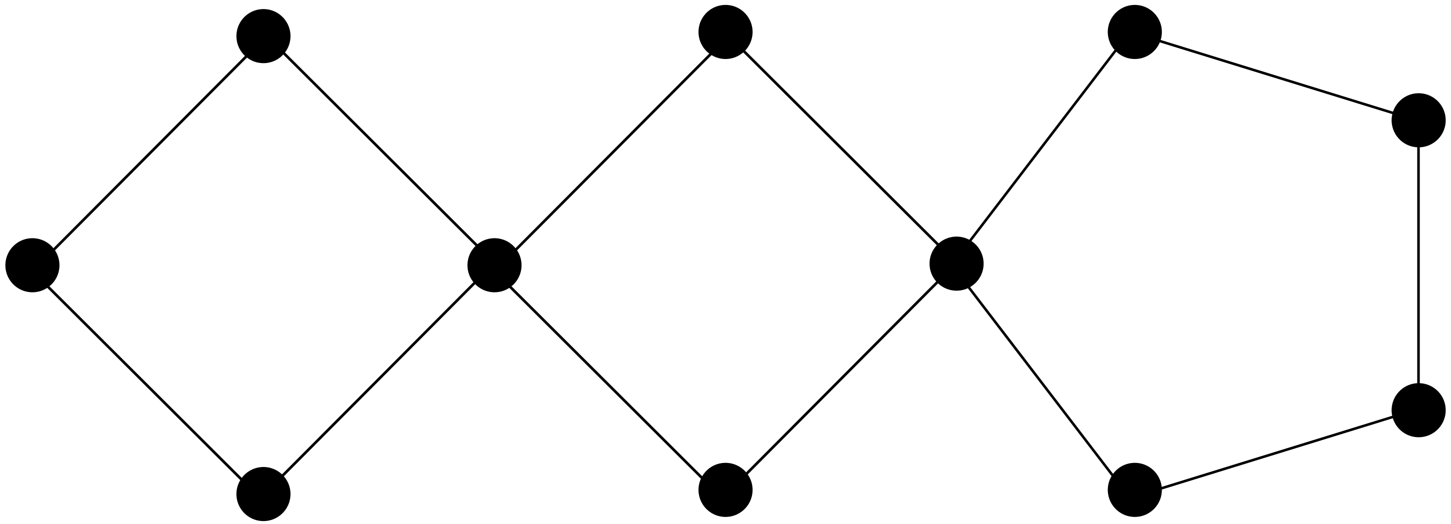


The minimum cardinality of a dominating set is called the domination number of G denoted by $\gamma(G)$.



A vertex subset S is a safe set of G if , for a component H of G-S and a component C of G[S], we have $|V(H)| \leq |V(C)|$ whenever there is an edge joining vertices between H and C.



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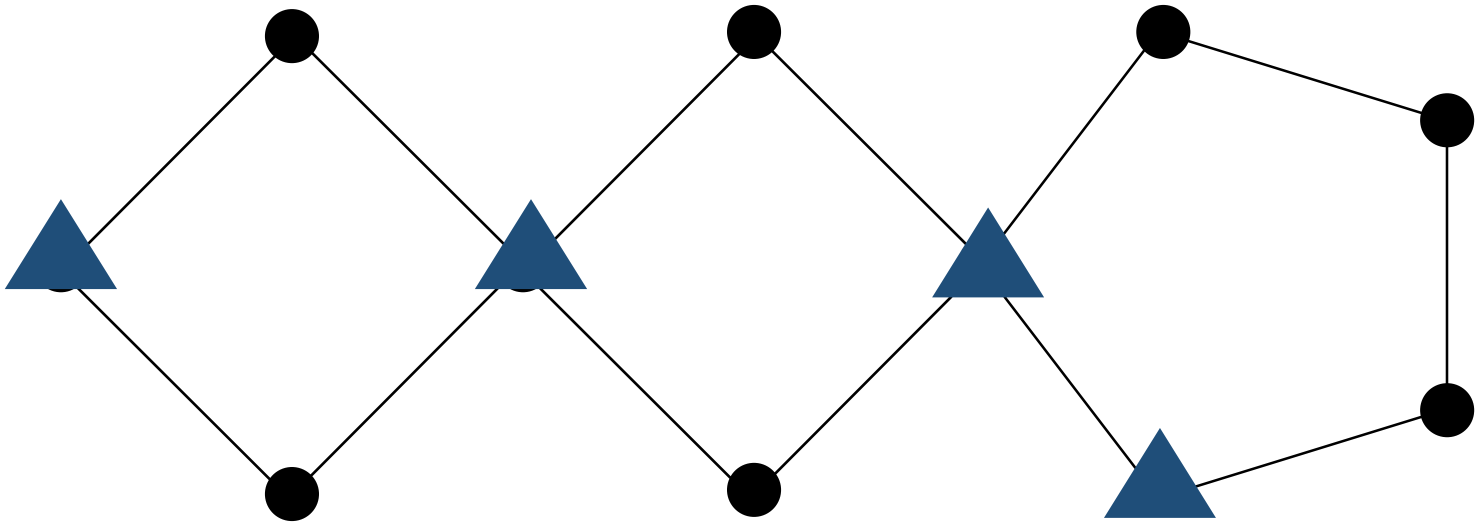

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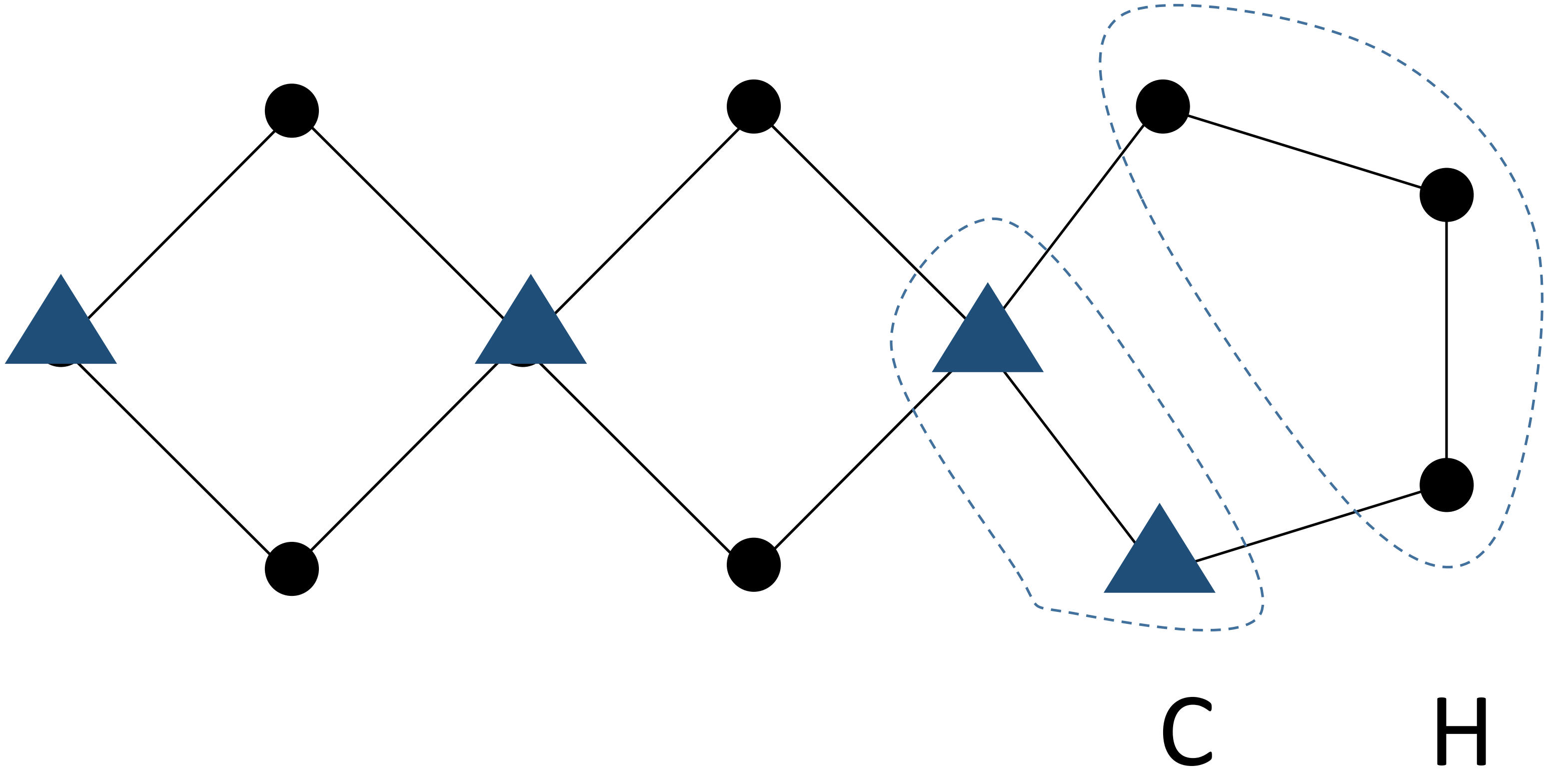

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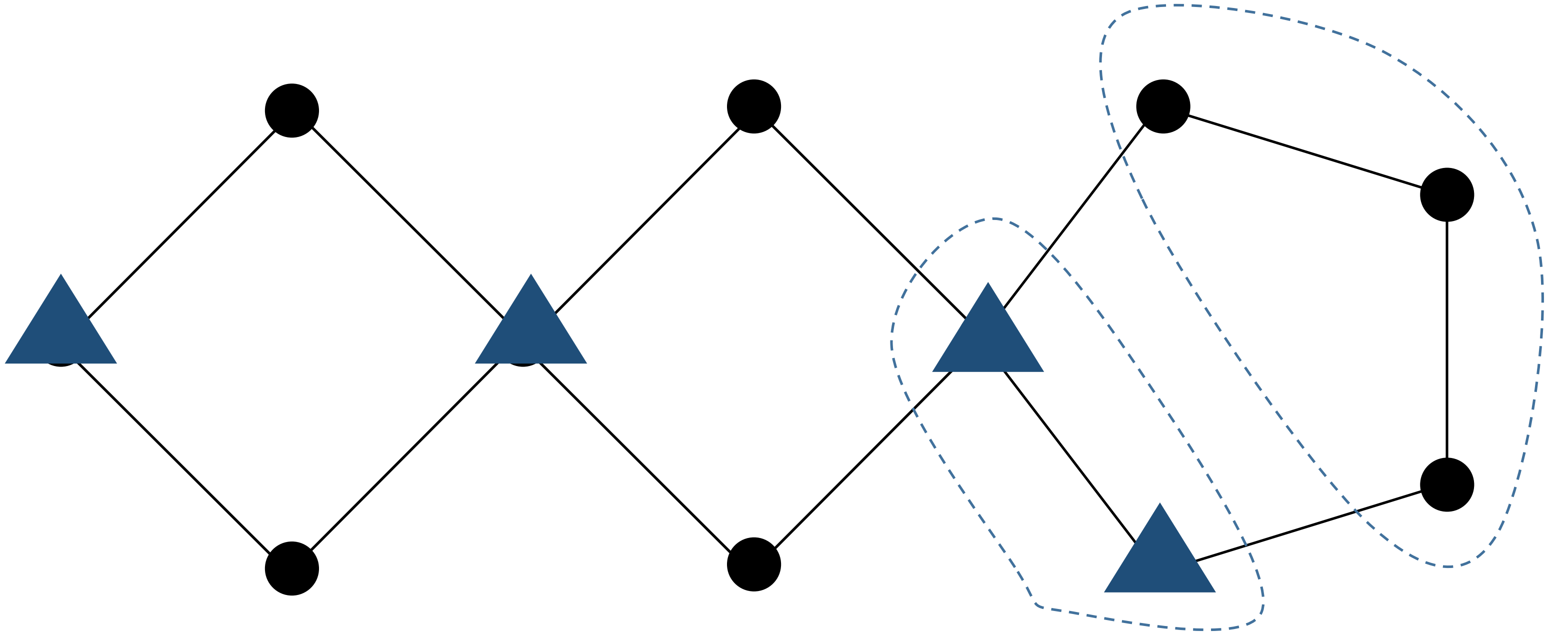

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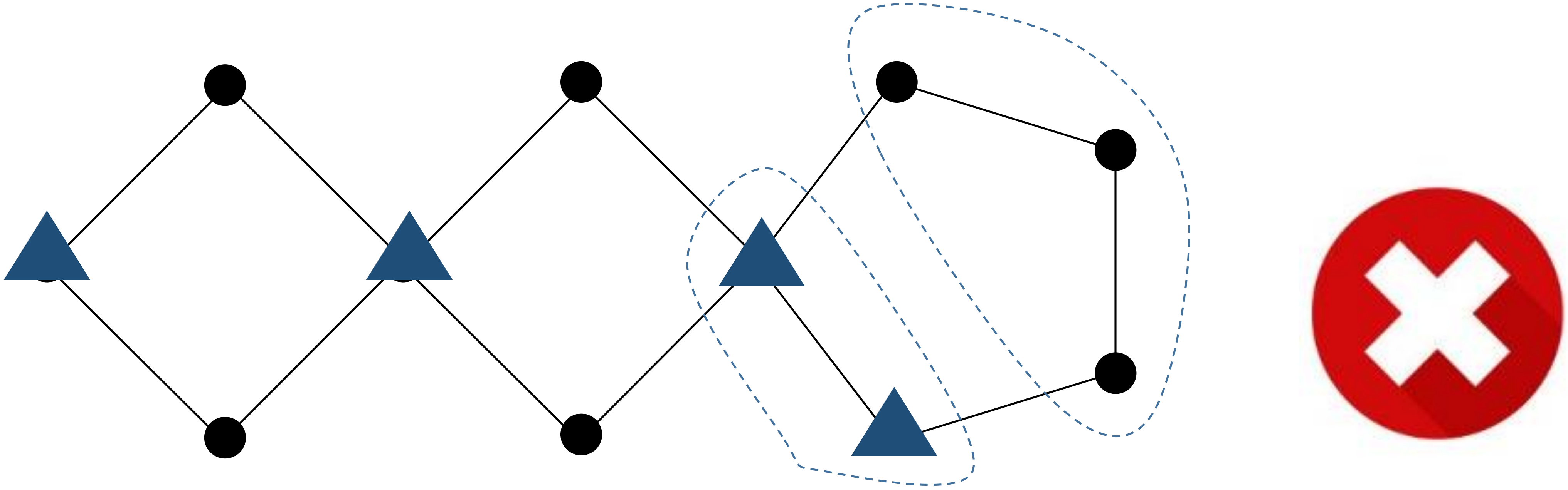

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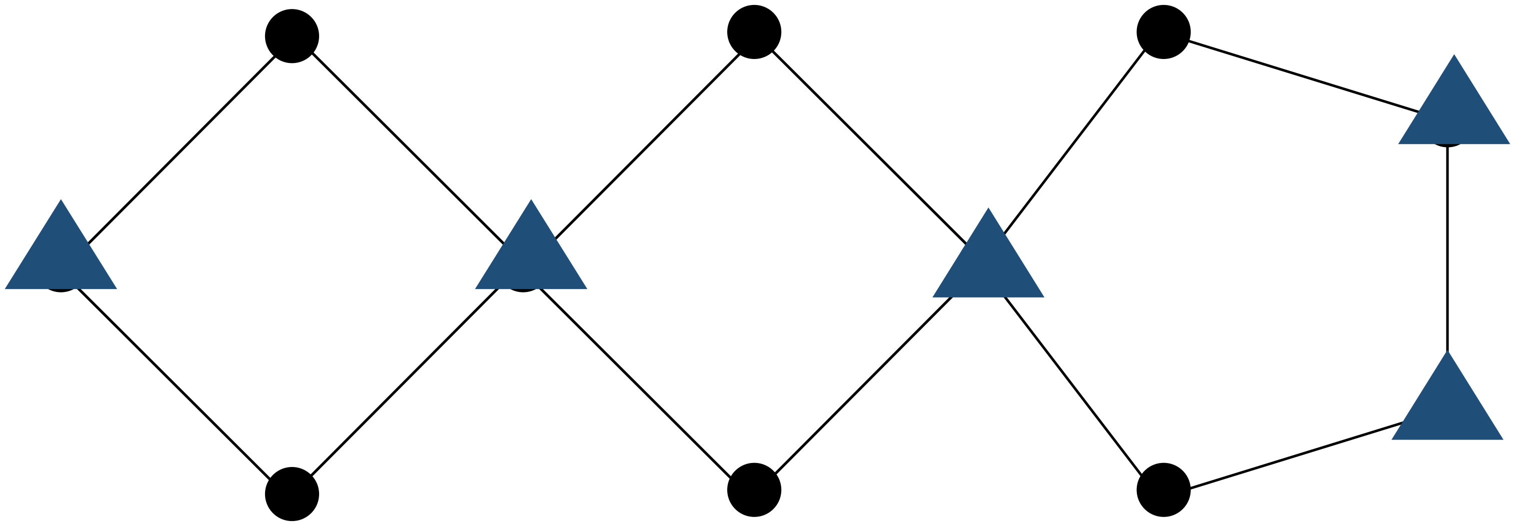

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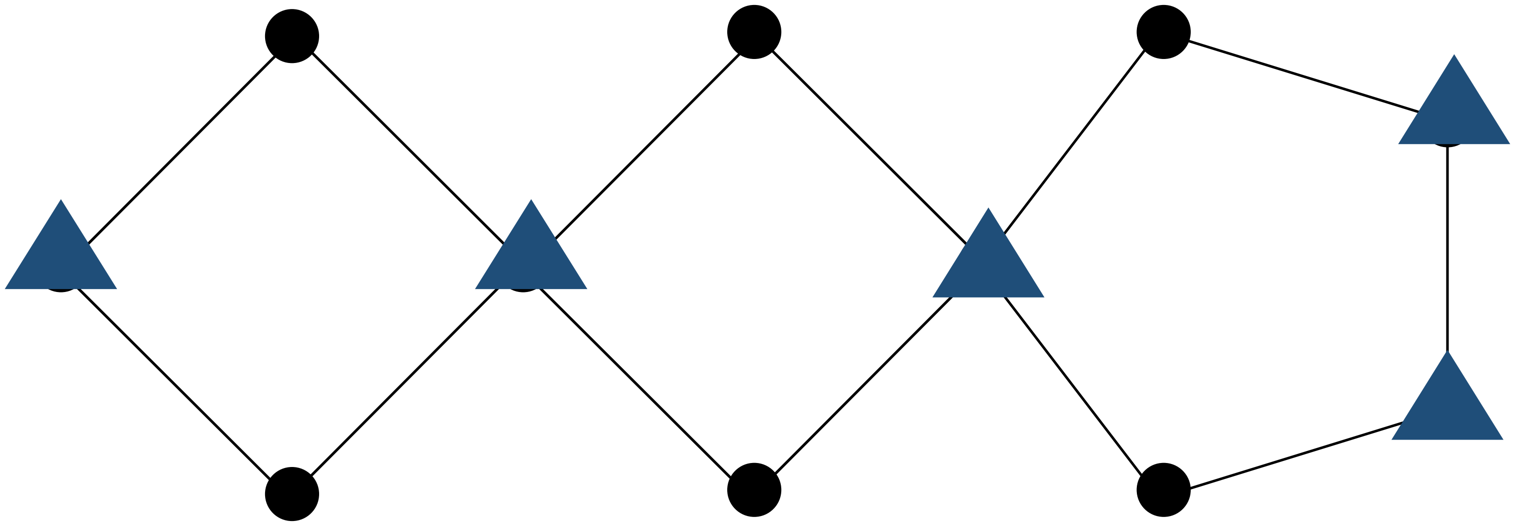

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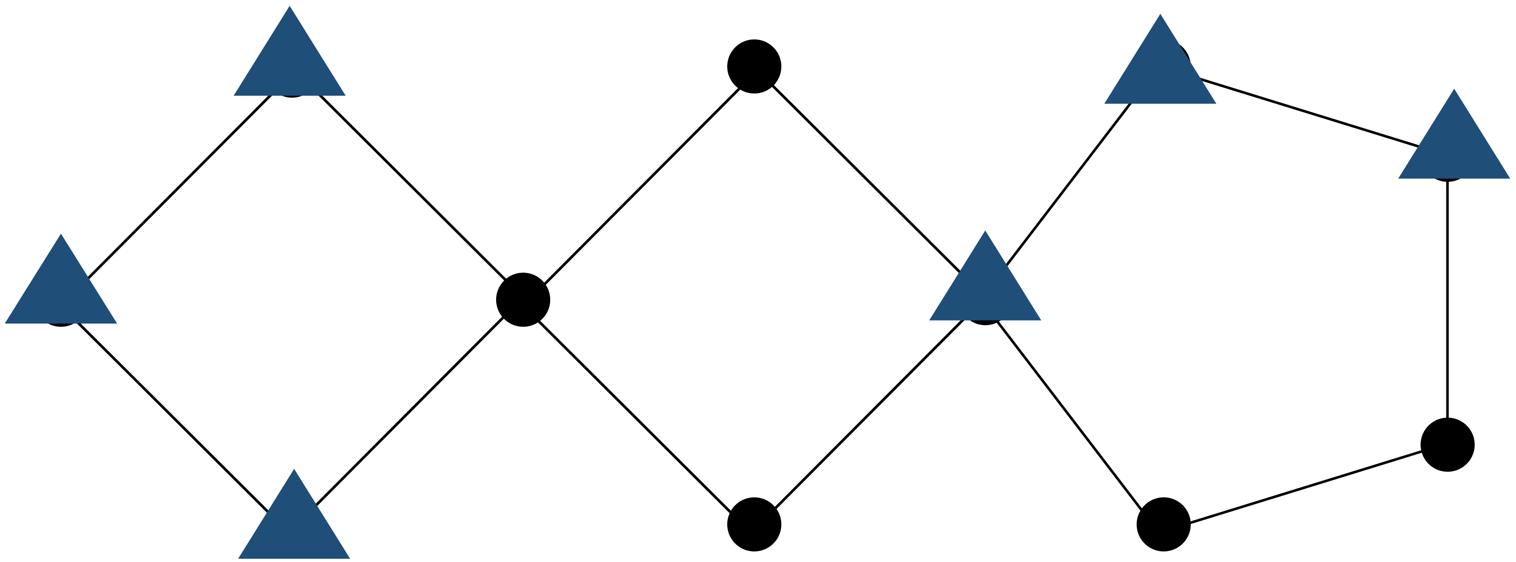

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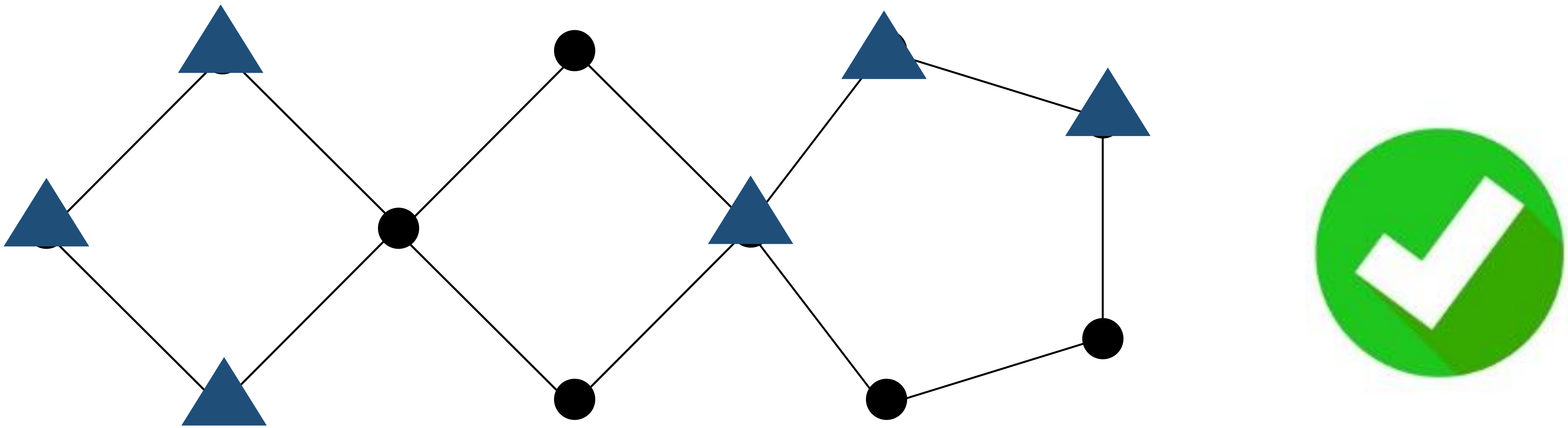

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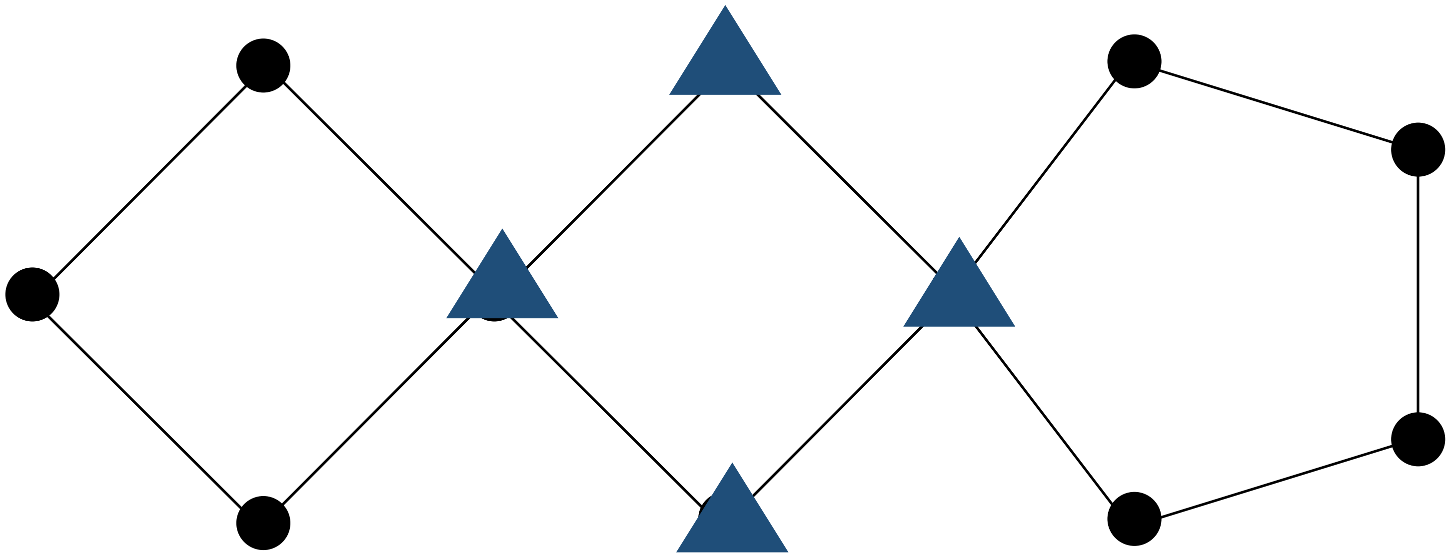
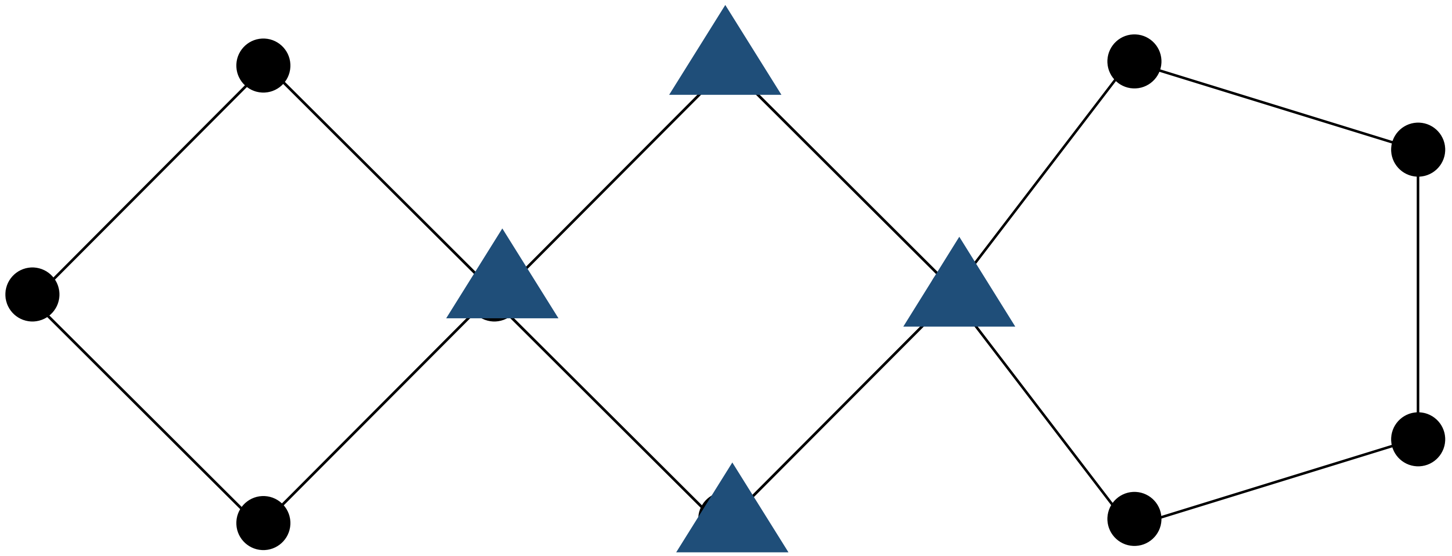


The minimum cardinality of a safe set of G is called the safe number of G and is denoted by s(G).



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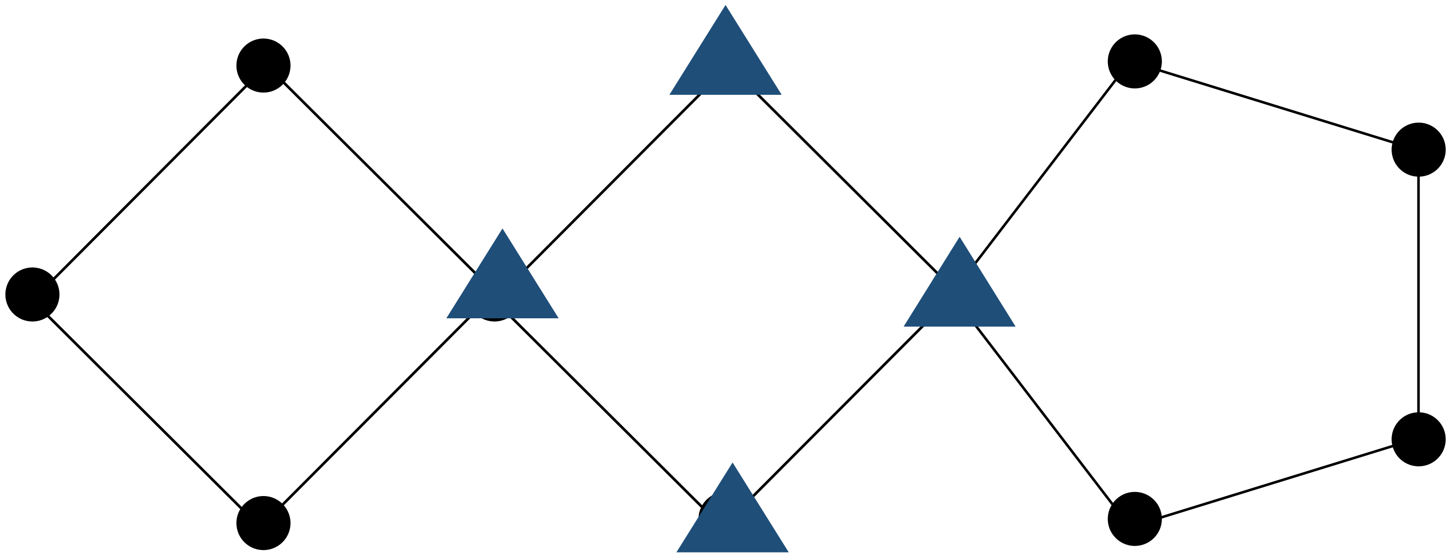


s(G) = 4

The historical origin of study of dominating sets in graphs began in 1862 when De Jaenisch studied the problem of finding the minimum number of queens that have to be placed on an $n \times n$ chessboard so that they dominate all the cells in the board.

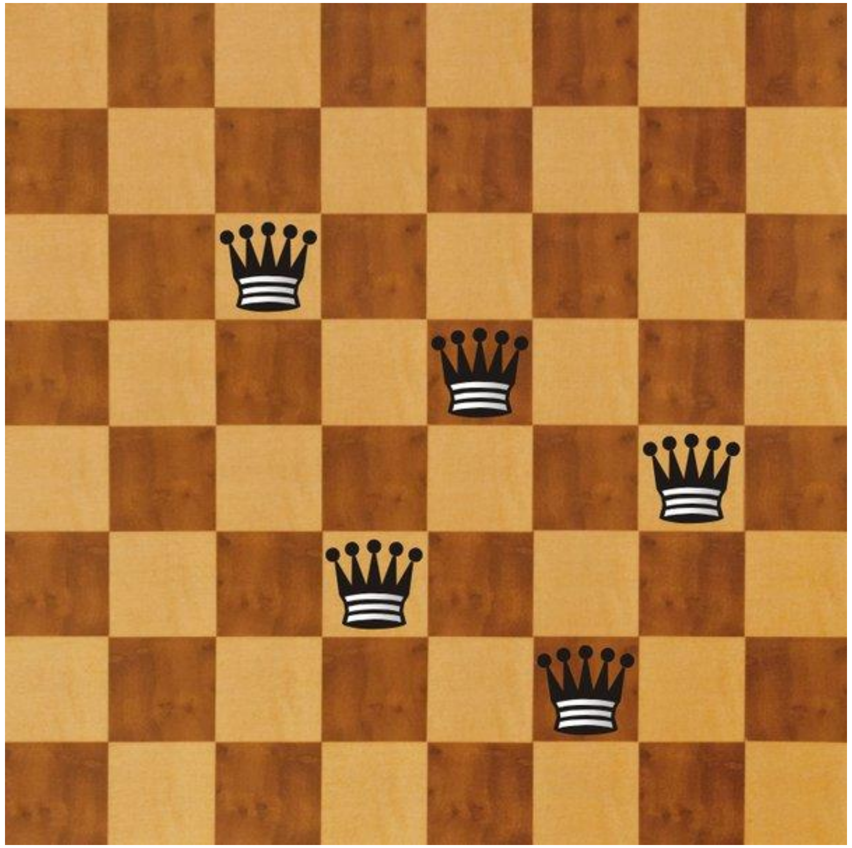
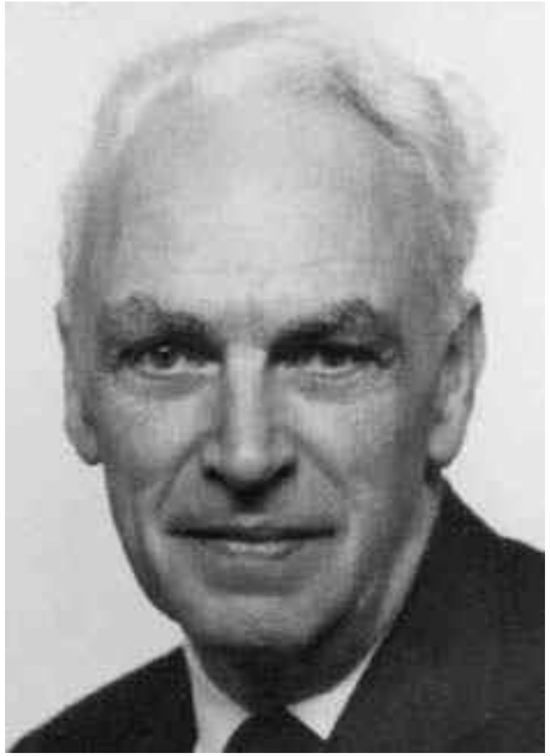


Photo from : www.quora.com/How many queens are required to cover every square in an 8*8 chessboard?

The concept of the domination number of a graph was introduced by Claude Berge in 1958 in his book on graph theory (the terminology used by him was 'coefficient of external stability').

Photo from: https://users.encs.concordia.ca/~chvatal/perfect/spgt.html

In 1962, Oystien Ore in his book on graph theory, used the names ‘dominating set’and ‘domination number’.



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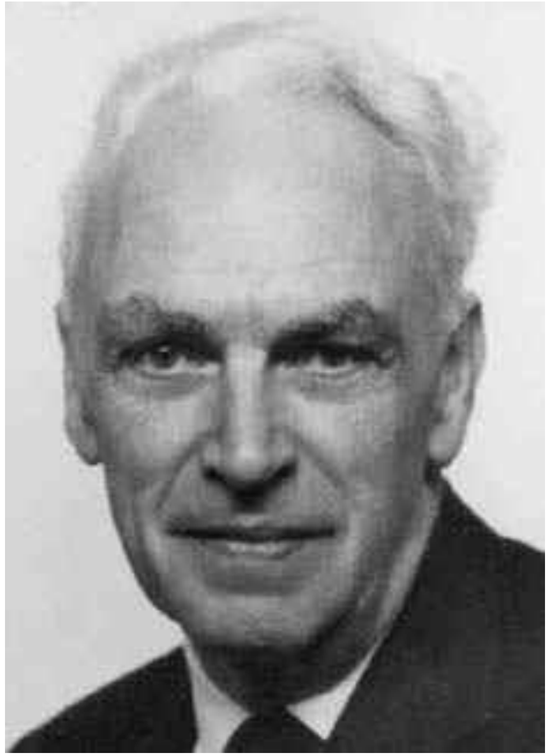

A thousand research papers related to domination have been published until now.

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S. Fujita and M. Furuya, Safe number and integrity of graphs, Discrete Applied Mathematics 247 (2018) 398–406.

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S. Ehard and D. Rautenbach, Approximating connected safe sets in weighted trees, Discrete Applied Mathematics 281 (2020) 216–223.

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Hosteins introduced a mixed integer linear programing formulation for safe sets

P. Hosteins, A compact mixed integer linear formulation for safe set problems, Optimization Letters 14 (2020) 2127-2148.

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2017

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In this talk, we give a guideline of the proof to show that

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Theorem 1 *Let G be a graph with the maximum degree Δ. Then*

$$f(\Delta) \leq s(G) \leq \lceil \frac{\gamma(G)(\Delta+1)}{2} \rceil$$

where $f(\Delta) = \frac{\gamma+6}{3}$ *when* $\Delta = 2$ *and* $f(\Delta) = \frac{\Delta^2-2\Delta-3+\sqrt{(2\Delta-\Delta^2+3)^2+4(3\Delta+\gamma(G))(\Delta-2)}}{2(\Delta-2)}$ *when* $\Delta \geq 3$.

By studying variations of safe set and domination in the "connected" aspect, we show that

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Theorem 2 *Let G be a connected graph. Then*

$$g(\Delta) \le s_c(G) \le \lceil \tfrac{\gamma_c(G)(\Delta-1)+2}{2} \rceil$$

where $g(\Delta) = \frac{\gamma_c(G)+2}{3}$ *when* $\Delta = 2$ *and* $g(\Delta) = \frac{\Delta-5+\sqrt{\Delta^2-2\Delta+4(\Delta-2)\gamma_c(G)+9}}{2(\Delta-2)}$ *when* $\Delta \ge 3$.

The upper bounds of both theorems are shown to be sharp. Further, we characterize all graphs satisfying the lower bound of each theorem.

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Observation *Let X be a vertex subset of t vertices of a graph G. If $G[X]$ is connected, then*

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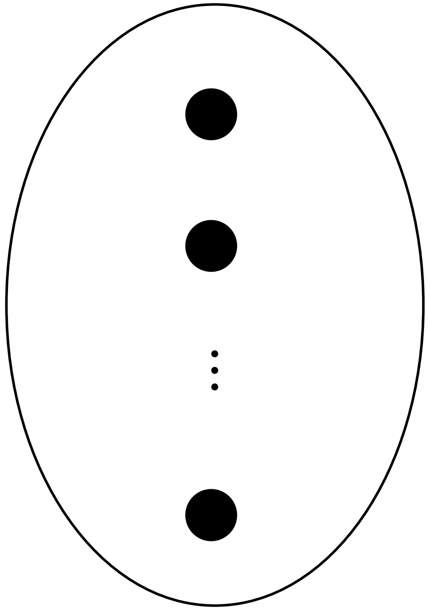

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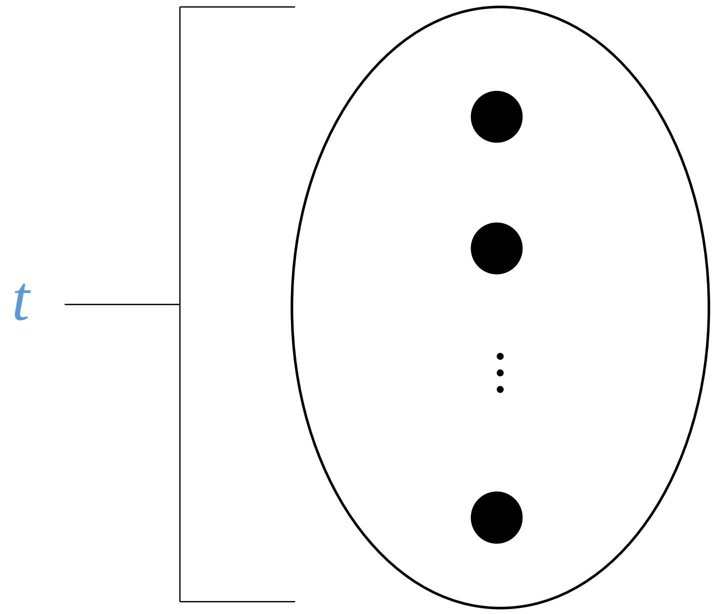

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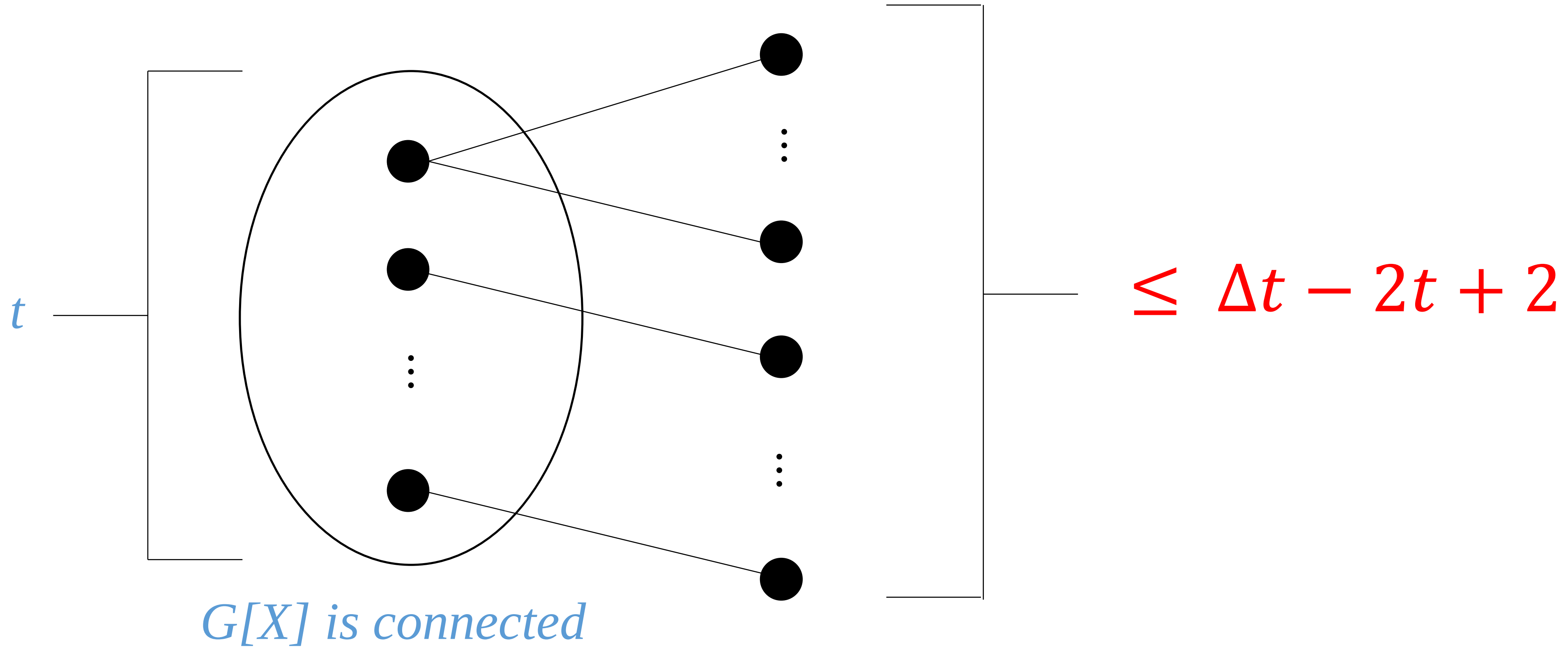

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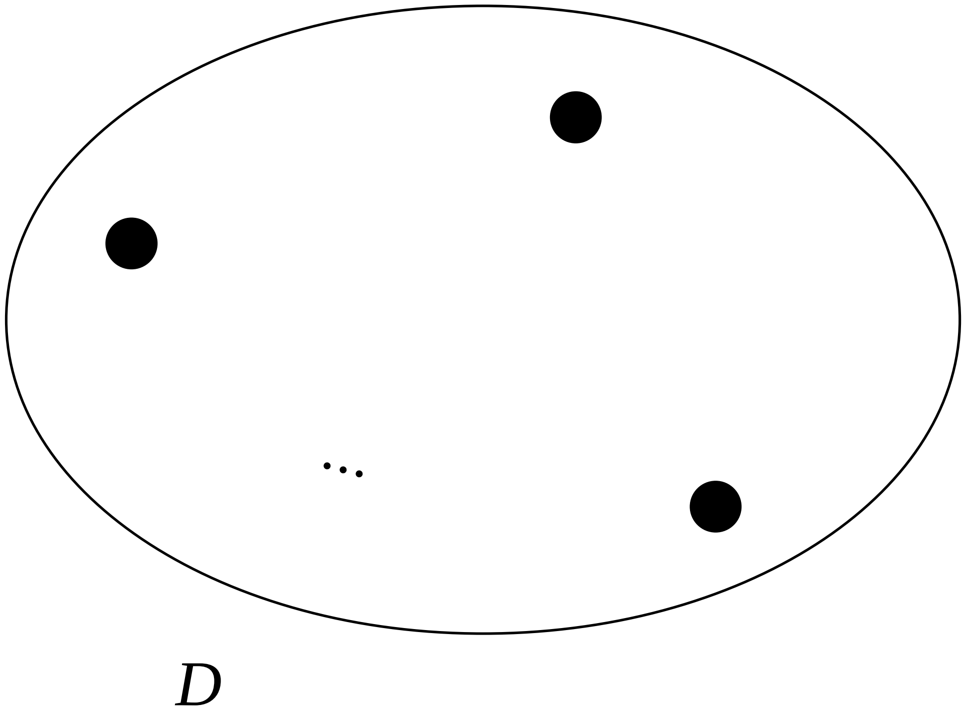

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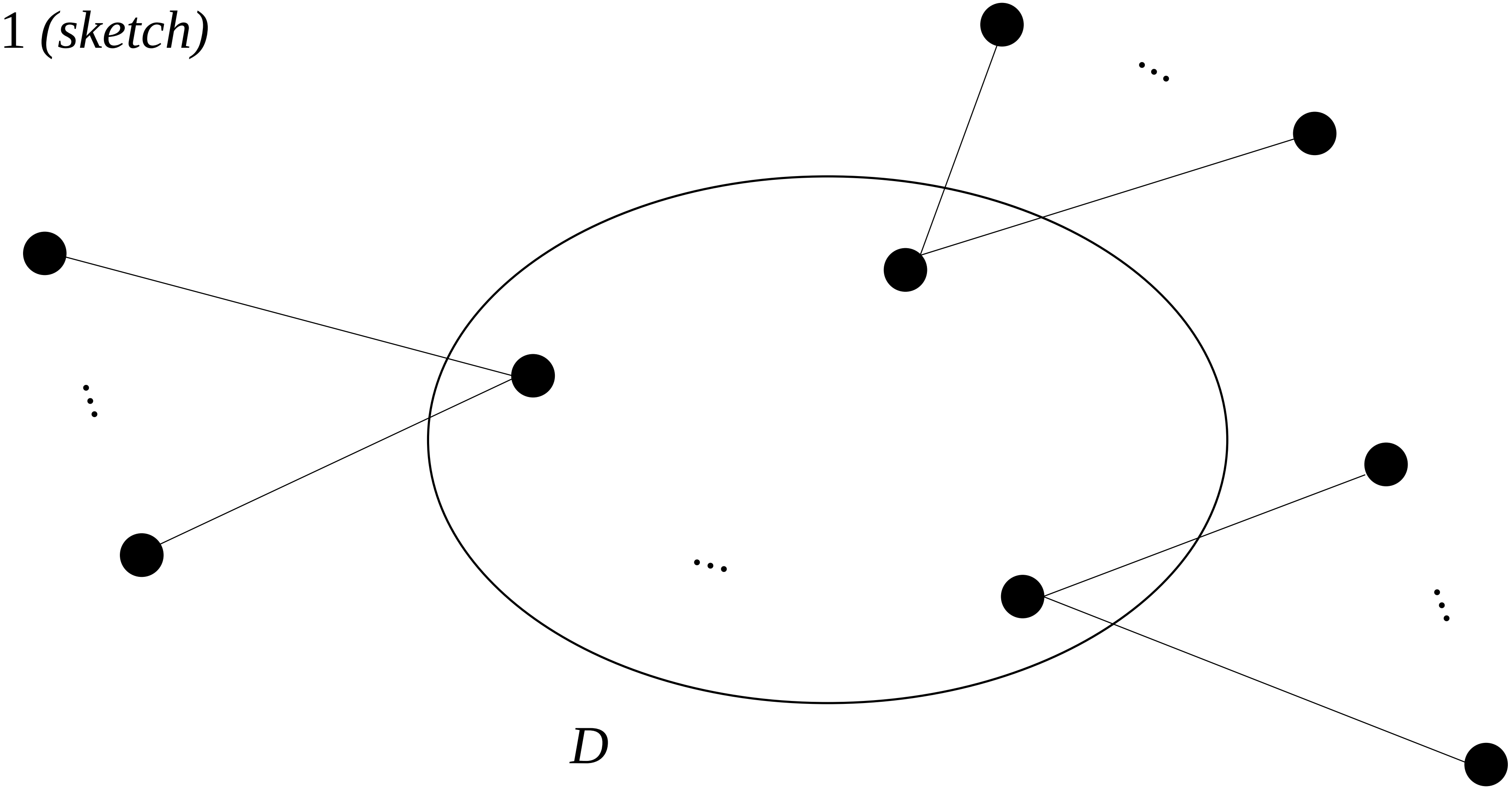

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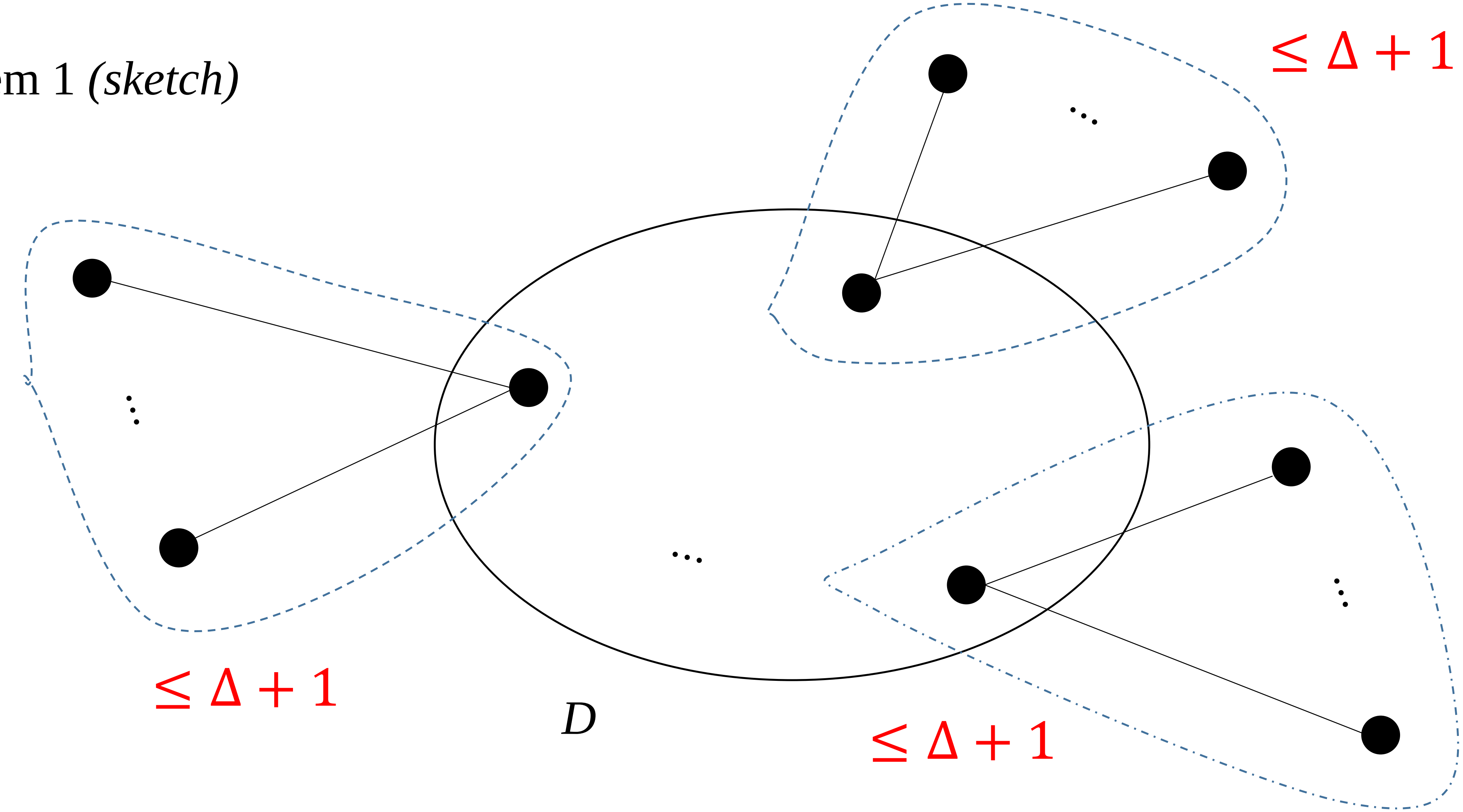

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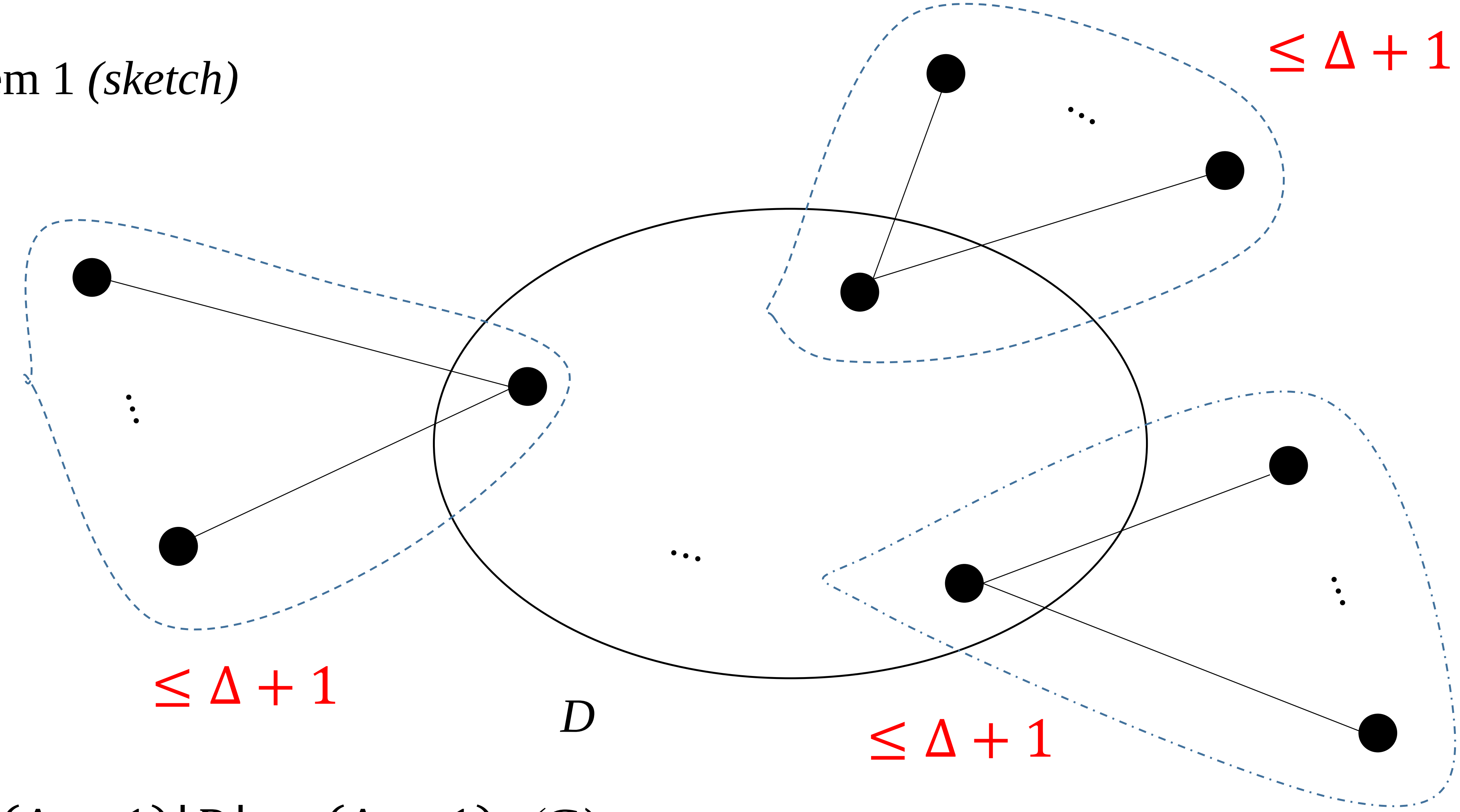

$\therefore |V(G)| \leq (\Delta + 1)|D| = (\Delta + 1)\gamma(G)$

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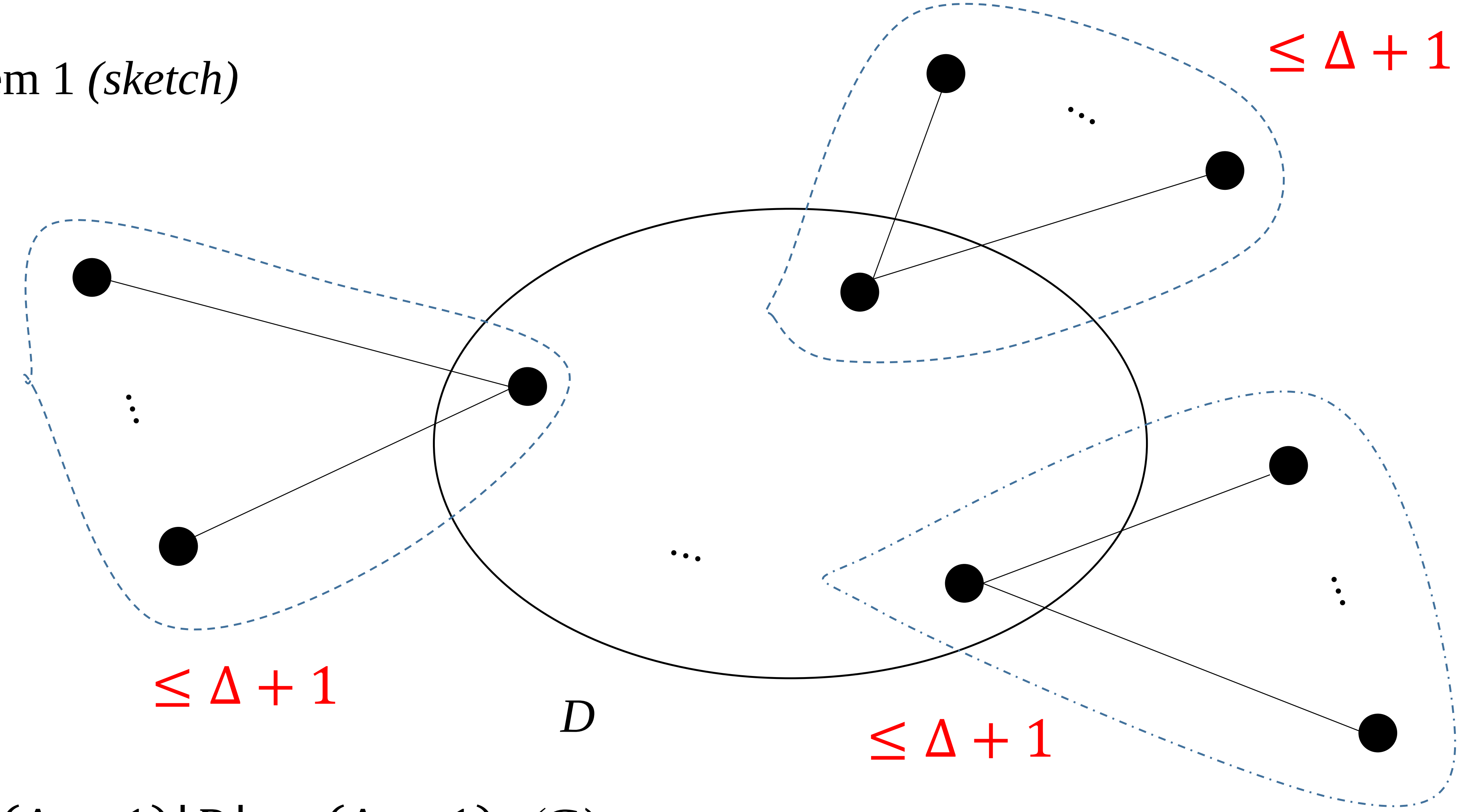

$\therefore\ |V(G)| \leq (\Delta + 1)|D| = (\Delta + 1)\gamma(G)$

Since $s(G) \leq \lceil |V(G)|/2 \rceil,$

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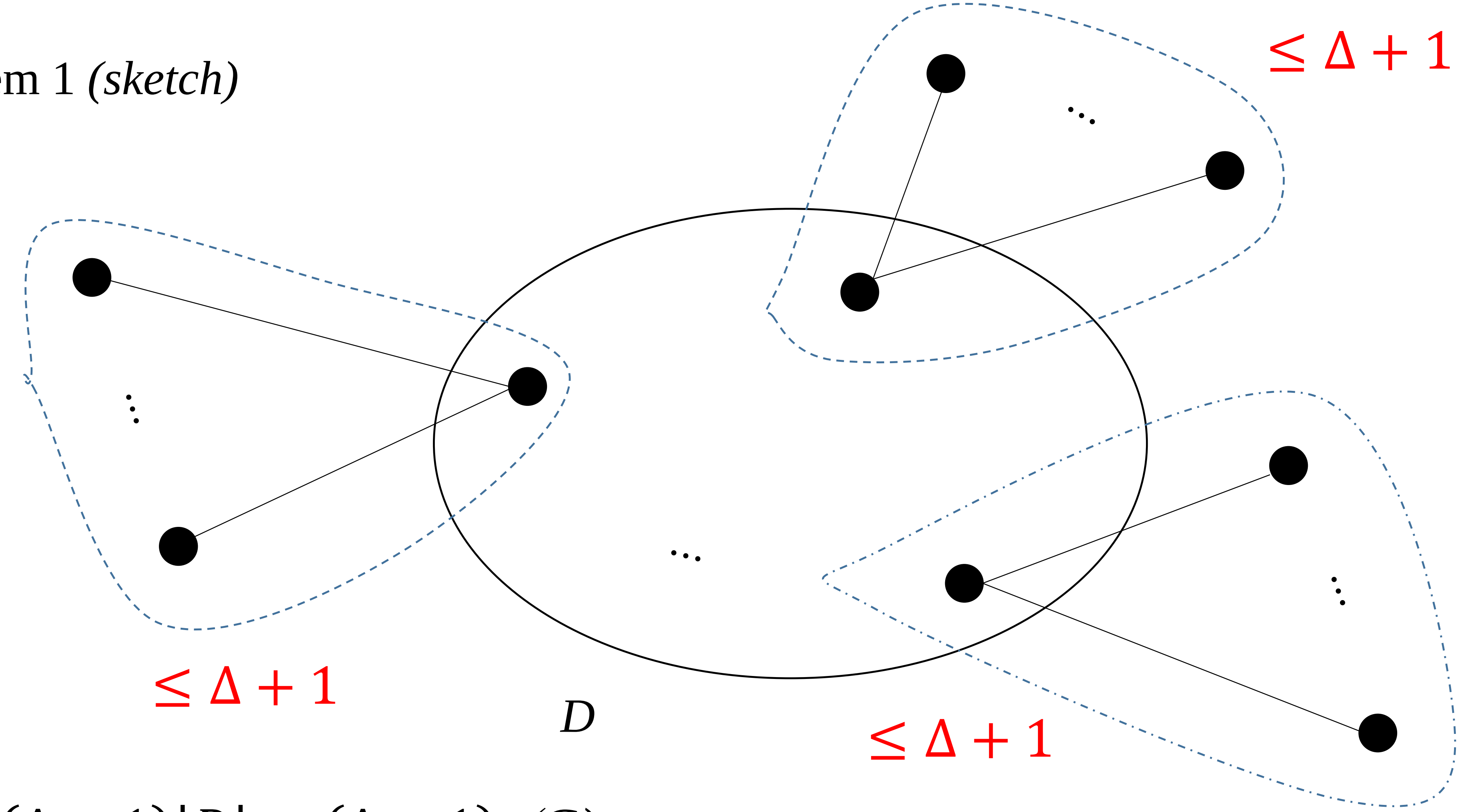
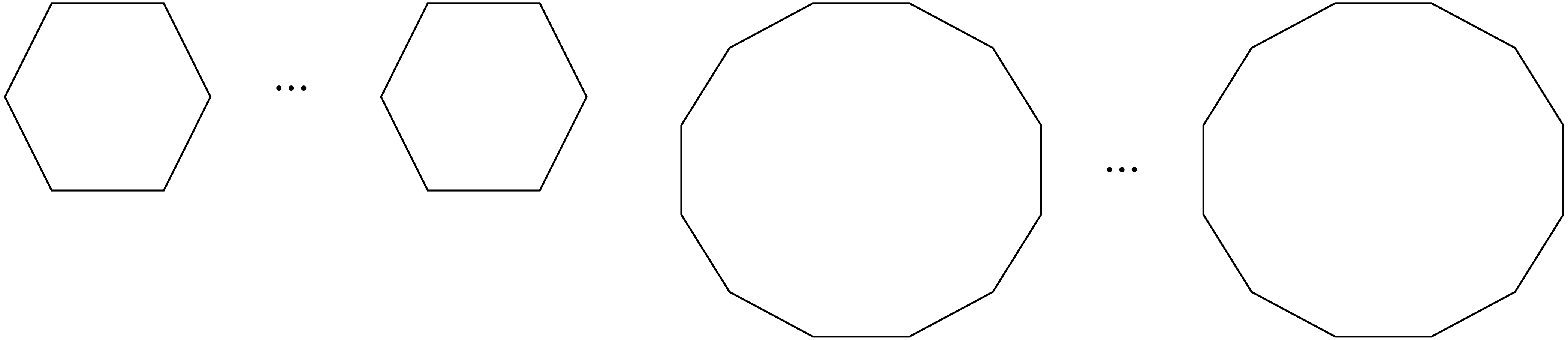


$\therefore\ |V(G)| \leq (\Delta + 1)|D| = (\Delta + 1)\gamma(G)$

Since $s(G) \leq \lceil |V(G)|/2 \rceil$, $s(G) \leq \lceil (\Delta + 1)\gamma(G)/2 \rceil$

Upper bound

Some extremal graphs satisfying the equality $s(G) = \lceil (\Delta + 1)\gamma(G)/2 \rceil$ *are*



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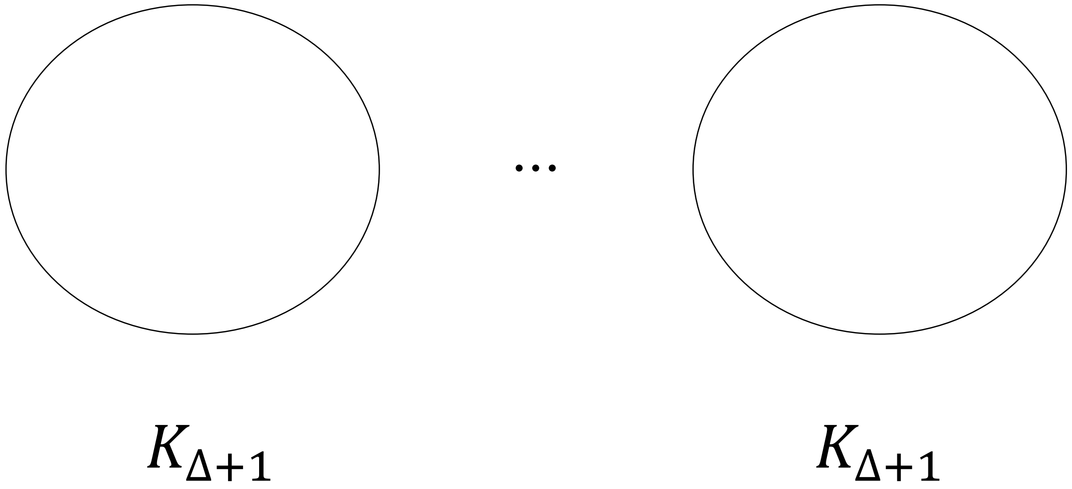

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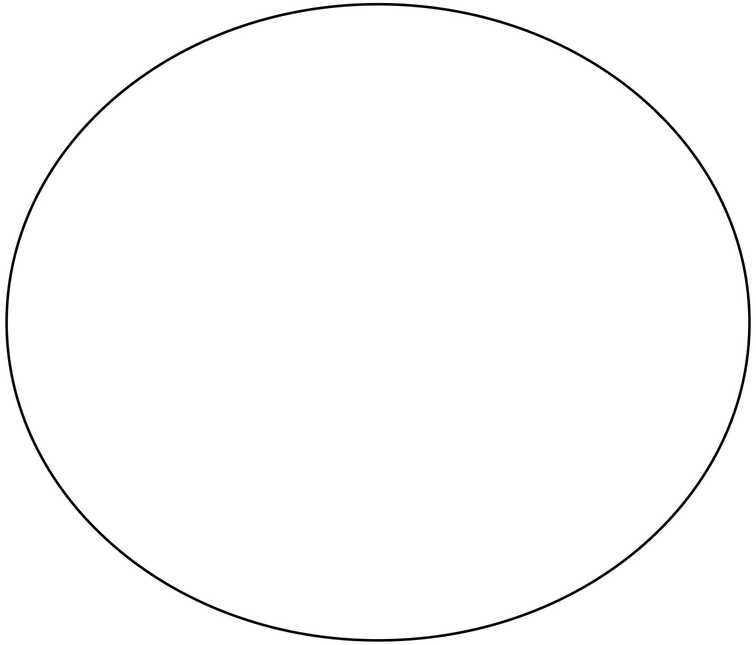
By Quadratic Formula, we would rather show that

$$\gamma(G) \leq (s(G)-\Delta)(\Delta s(G)-2s(G)+3).$$



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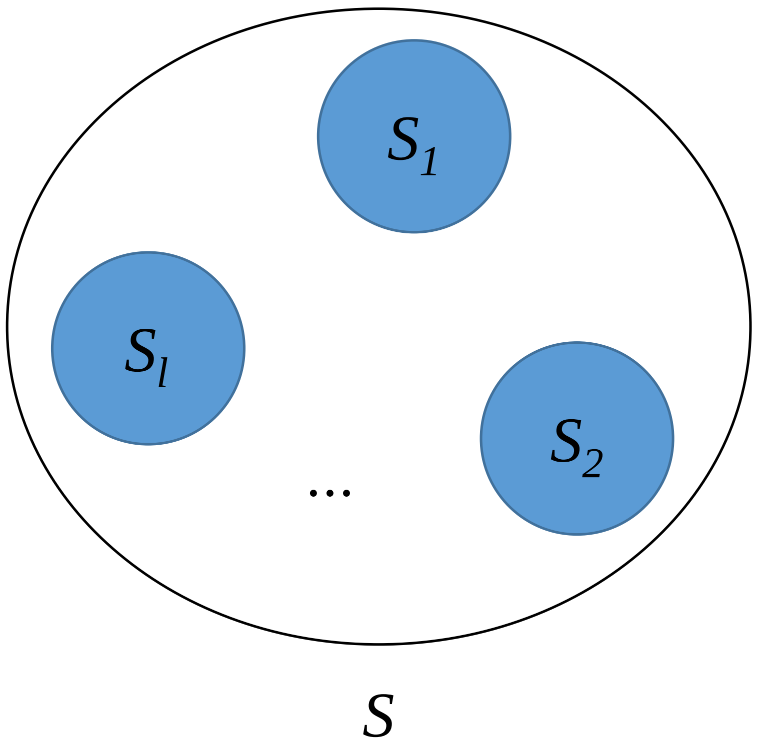
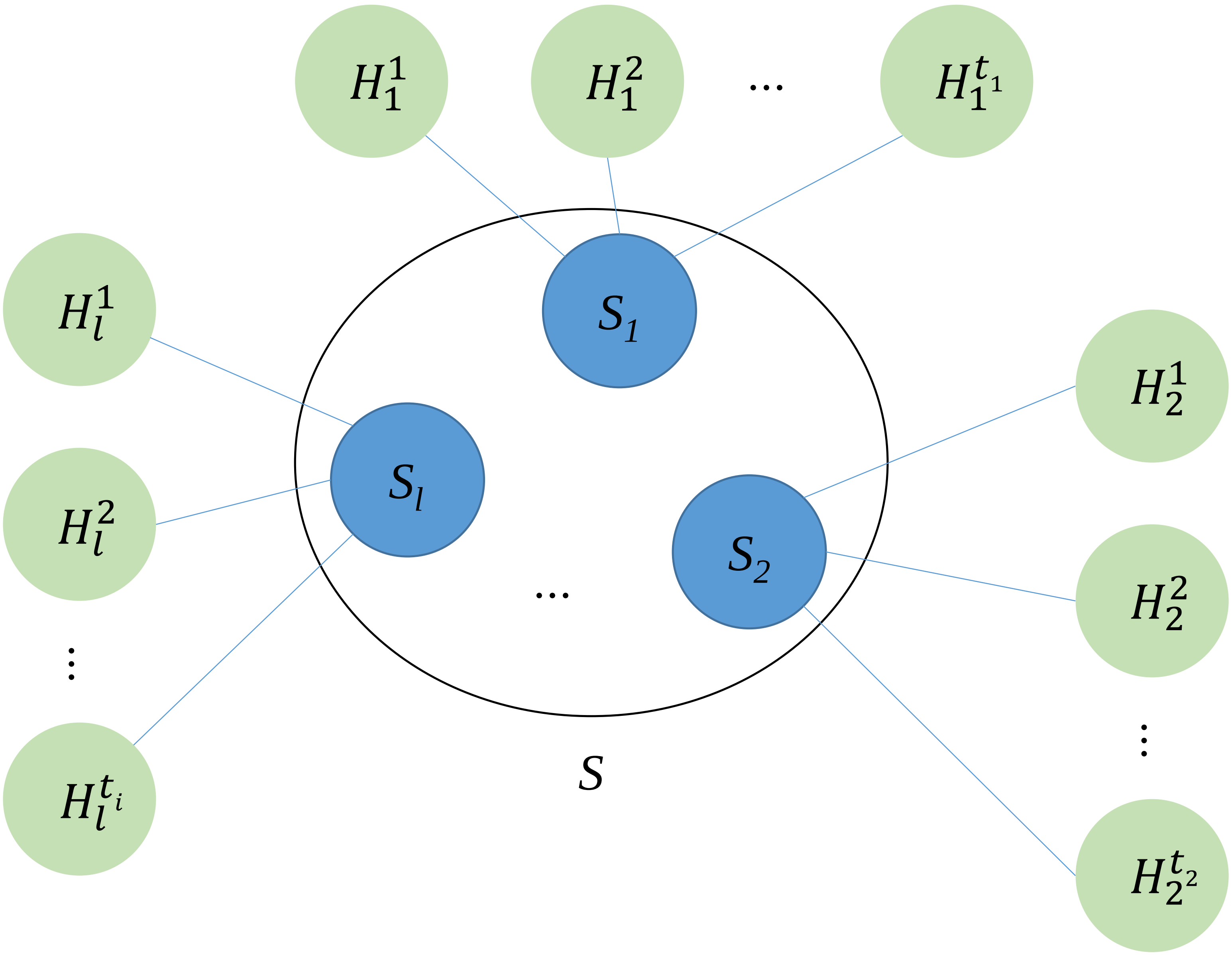
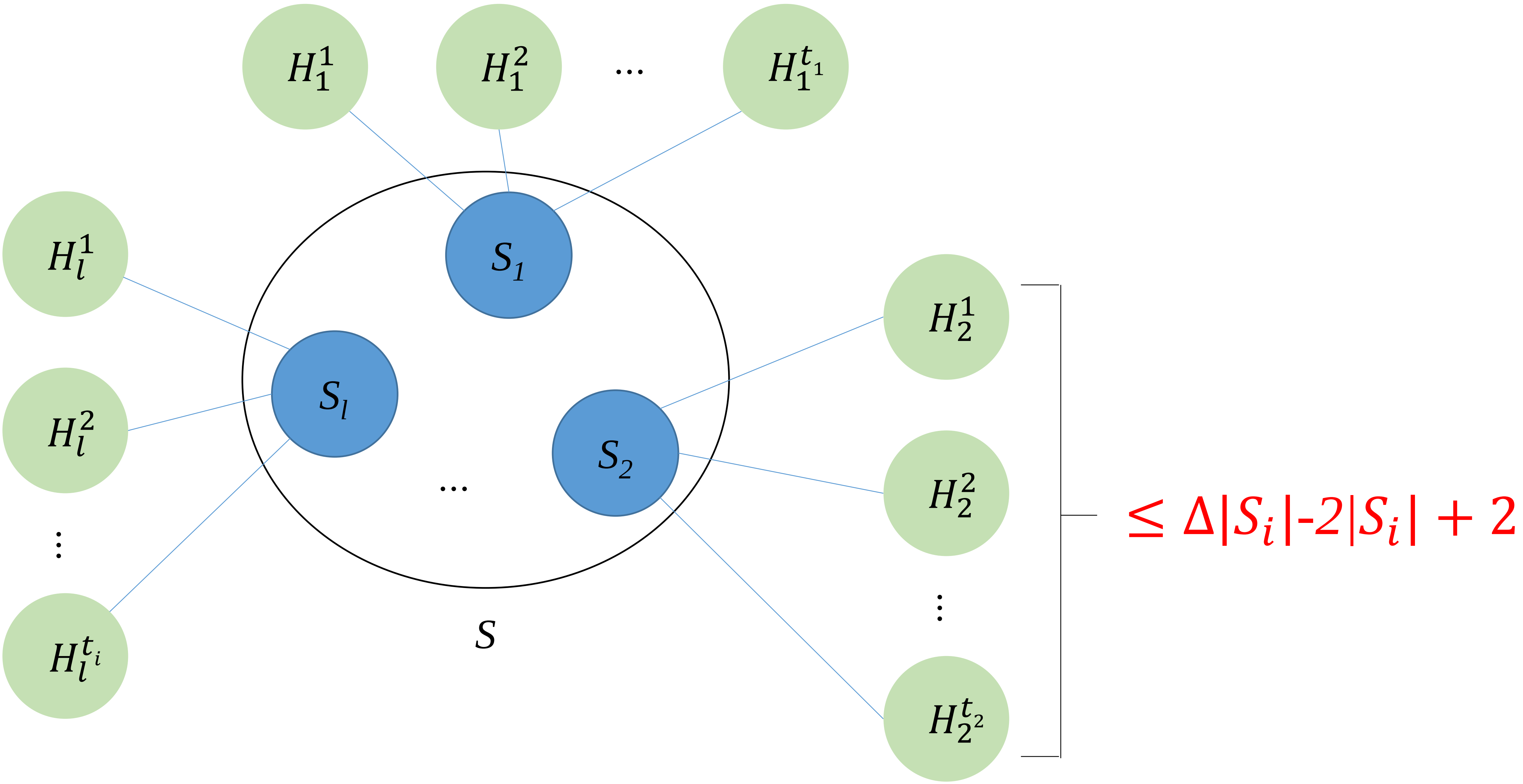

S_1
S_l
S_2
...
S

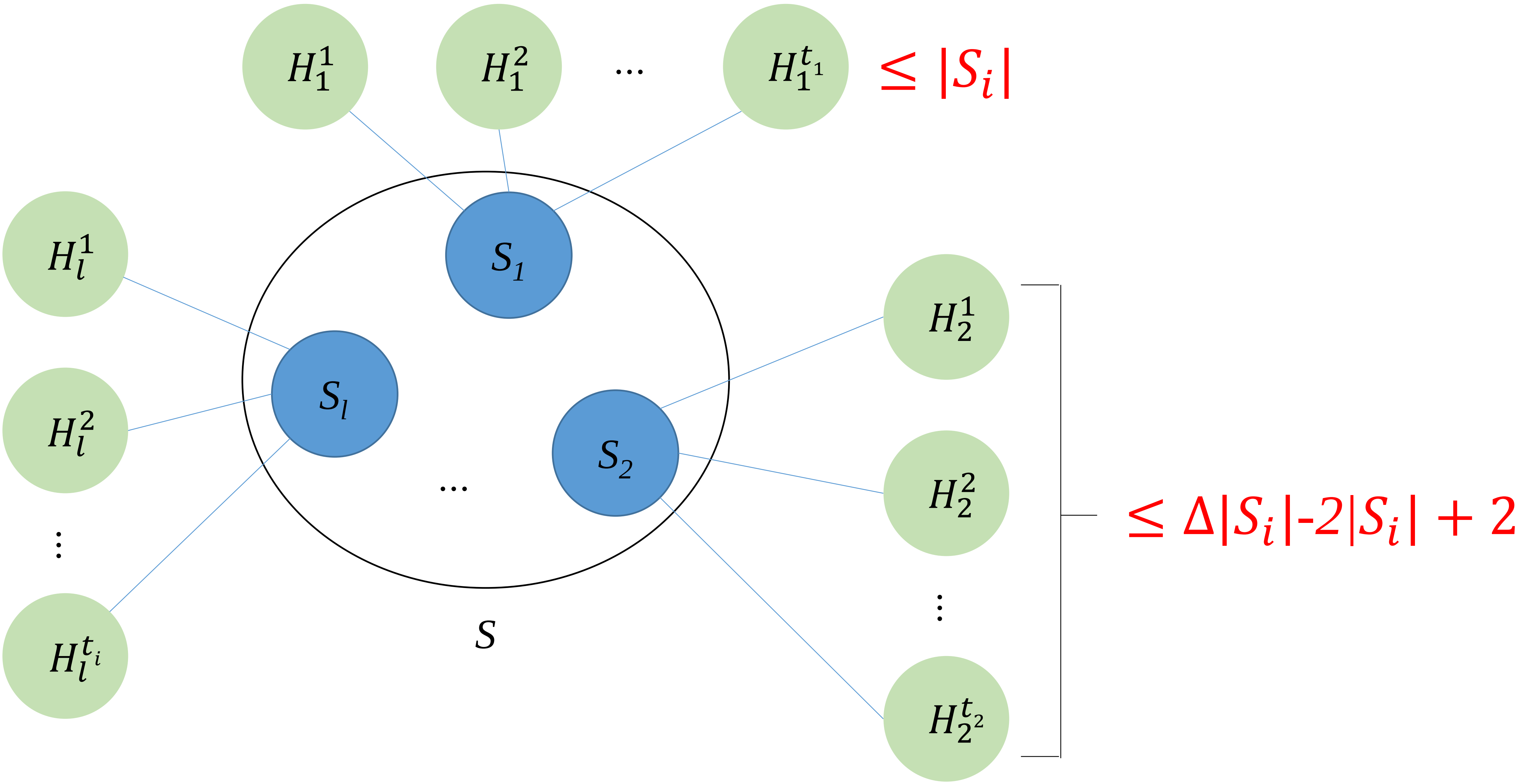


H_1^1
H_1^2
...
$H_1^{t_1}$
H_l^1
S_1
H_2^1
S_l
H_l^2
S_2
...
H_2^2
$H_l^{t_i}$
S
$H_2^{t_2}$

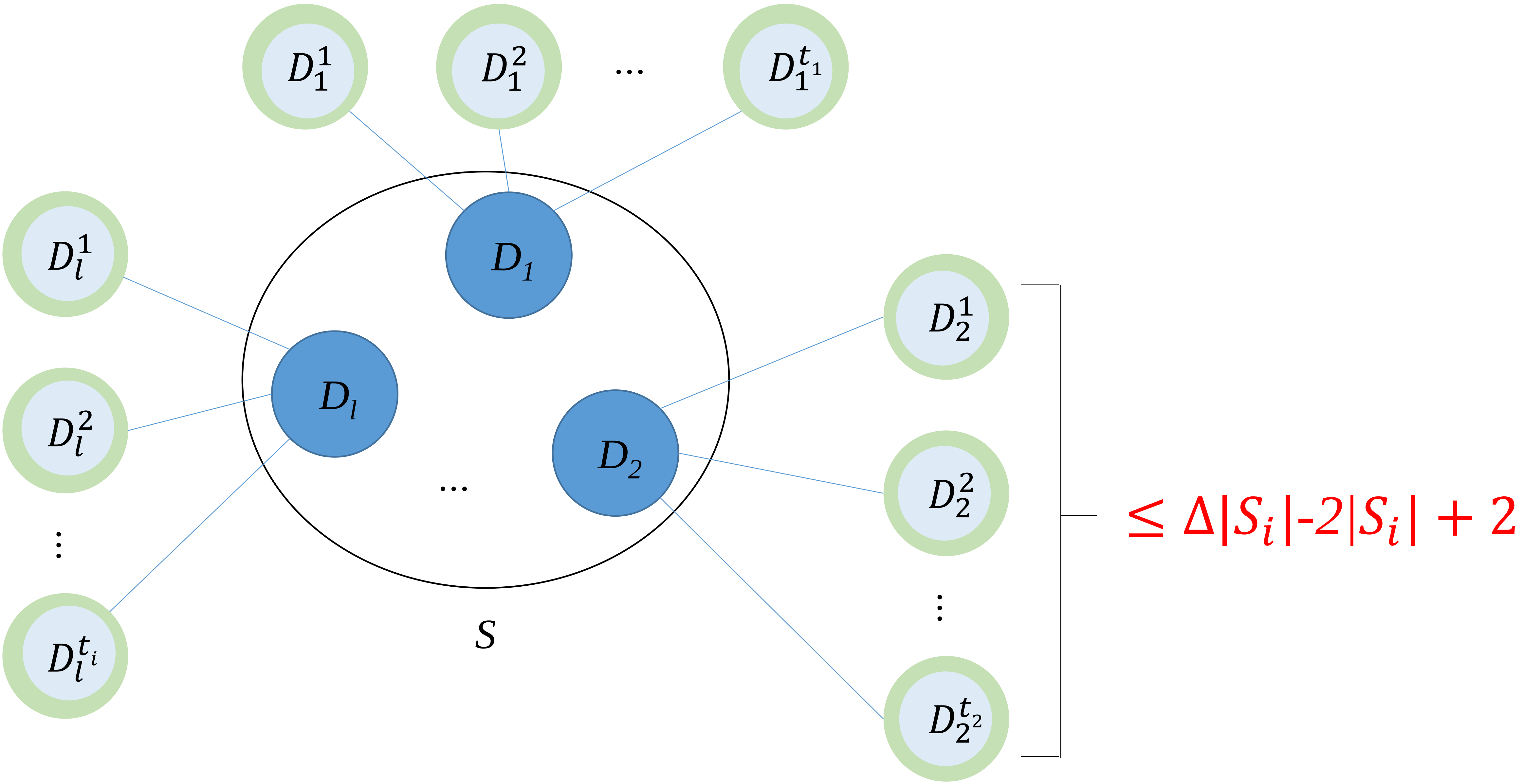
By the observation, we have



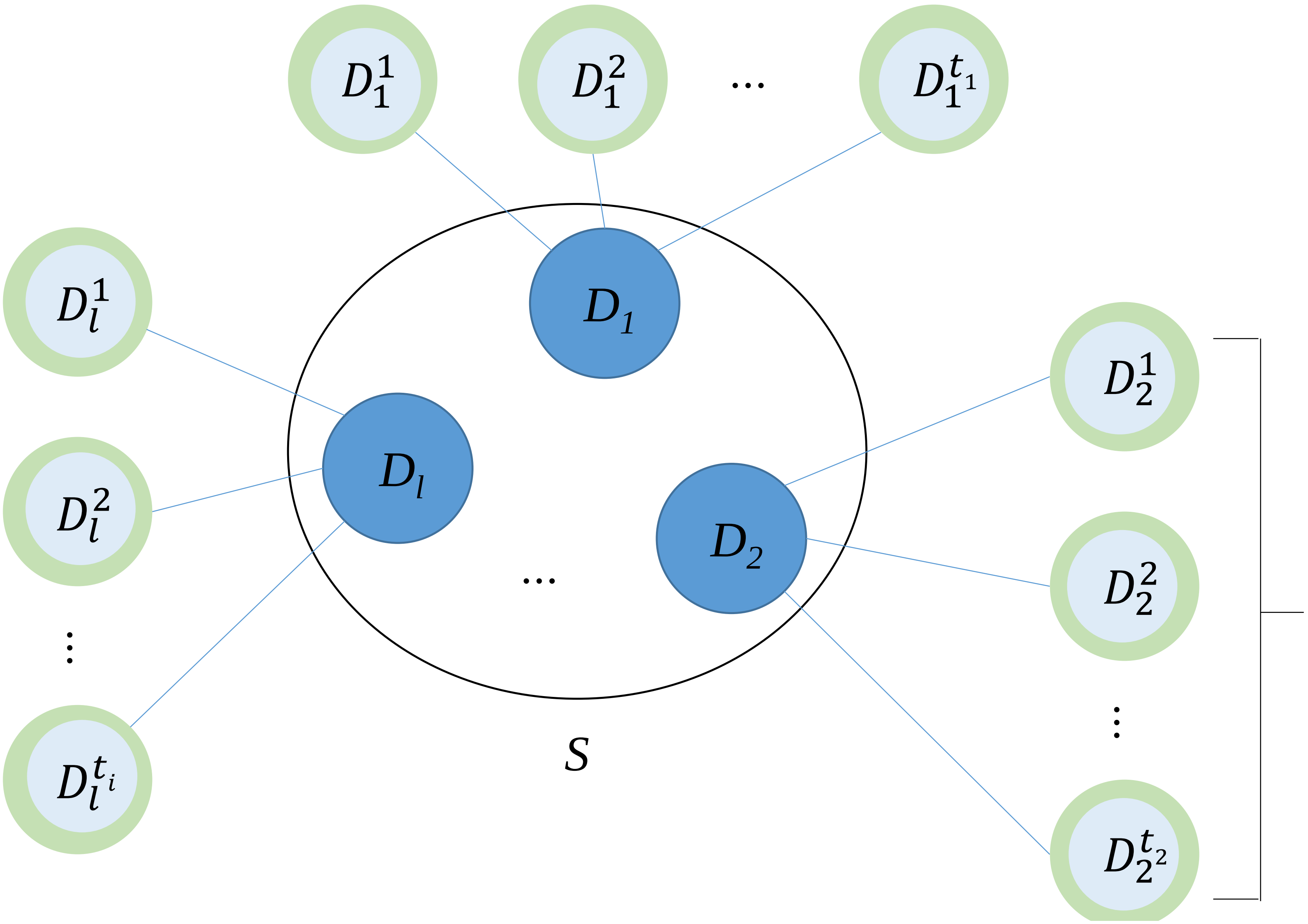
Because S is a safe set, we have



Now, each part of the graph has its own dominating set



By the result of Berg, we have $|D_i^j| \le |S_i| - \Delta$



$\le \Delta|S_i|$-$2|S_i| + 2$

C. Berge, Theory of Graphs and its Applicationa, Methuen, London, 1962.

Then, the union of all the dominating sets gives

$$\gamma(G) \leq (s(G) - \Delta)(\Delta s(G) - 2s(G) + 3).$$

The characterization of graphs achieving the equality

$$\gamma(G) = (s(G) - \Delta)(\Delta s(G) - 2s(G) + 3).$$

The class $\mathcal{A}_1(\Delta, s)$.

For given positive integers $s > \Delta$, a graph H in this class has order s and contains a vertex x of maximum degree Δ. Further, H has the following properties.

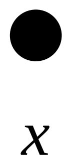
(*a*) $\gamma(H) = s - \Delta$

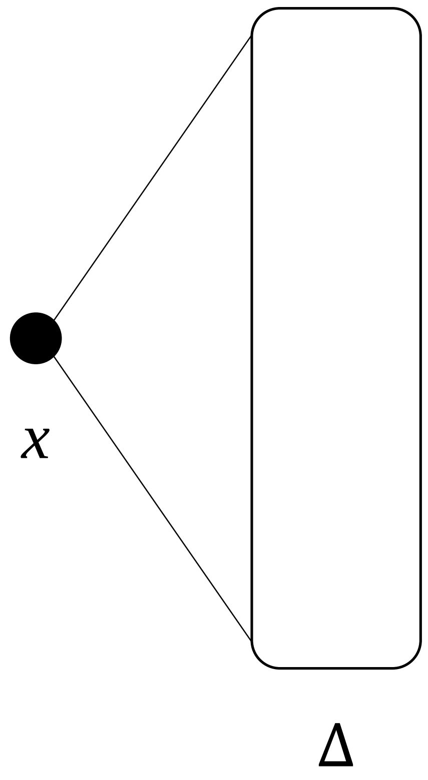
(*b*) $H - N_H[x]$ is independent,

(*c*) every vertex in $N_H(x)$ is adjacent to at most one vertex in $H - N_H[x]$,

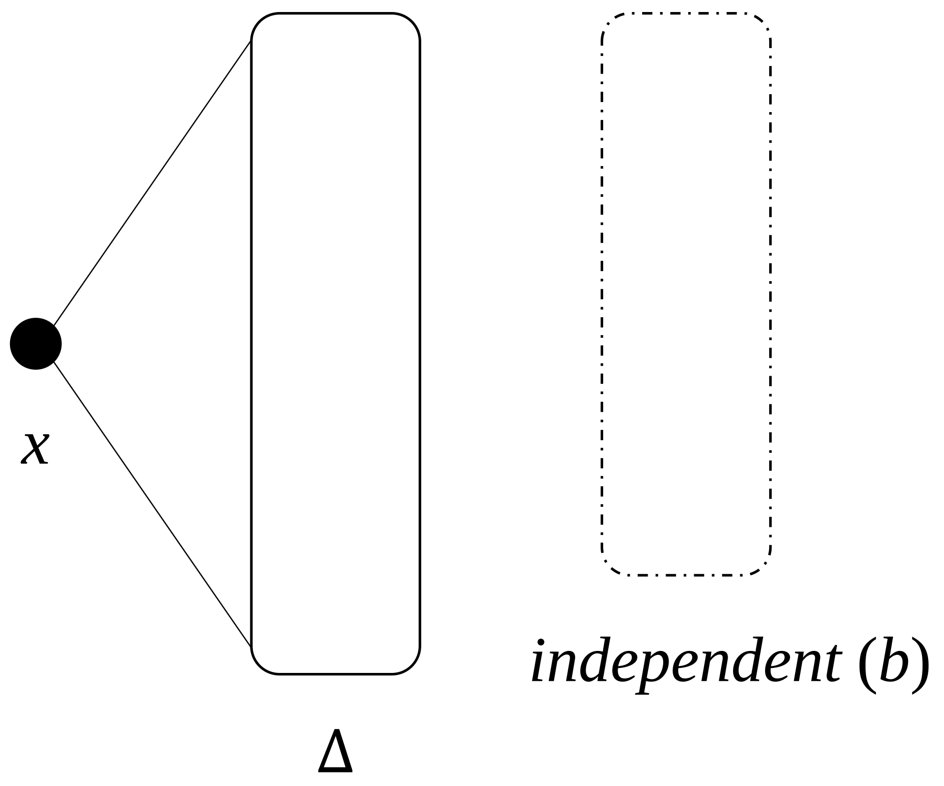
(*d*) $|V(H) - N_H[x]| < \Delta$ and

(*e*) H has at least one bad non-critical vertex or at least one non-critical vertex (this vertex is not x by the characterization of graphs H satisfying $\gamma(H) = |V(H)| - \Delta(H)$ which we always have a γ-set $\{x\} \cup (V(G) - N_G[x])$ containing a vertex of maximum degree x).

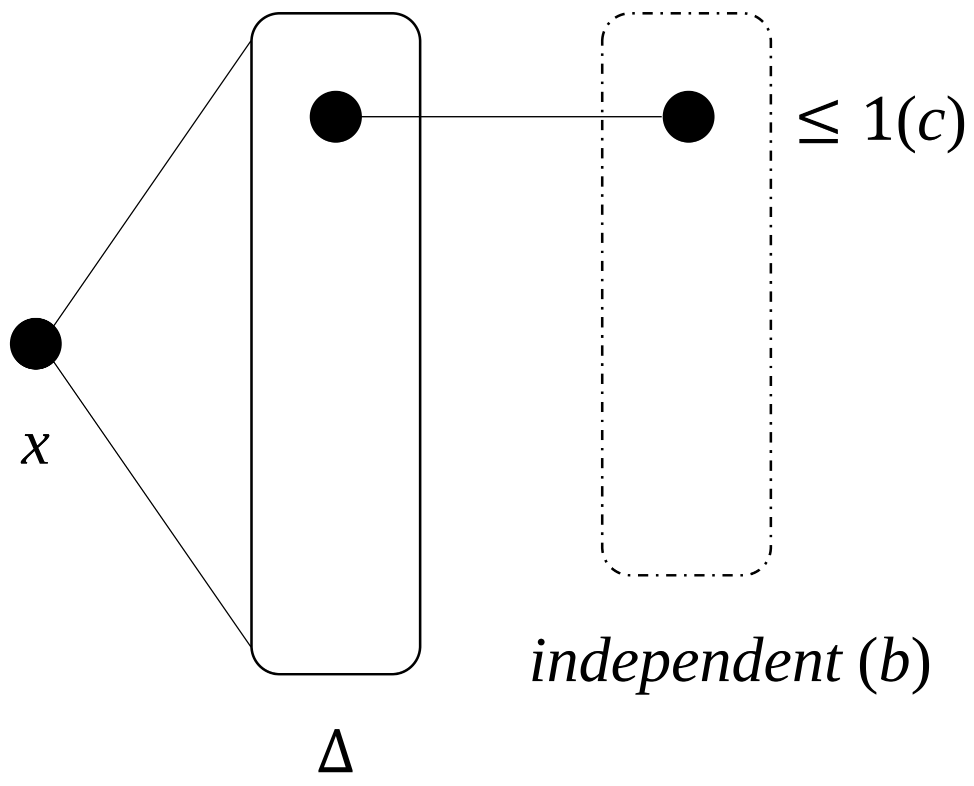

x



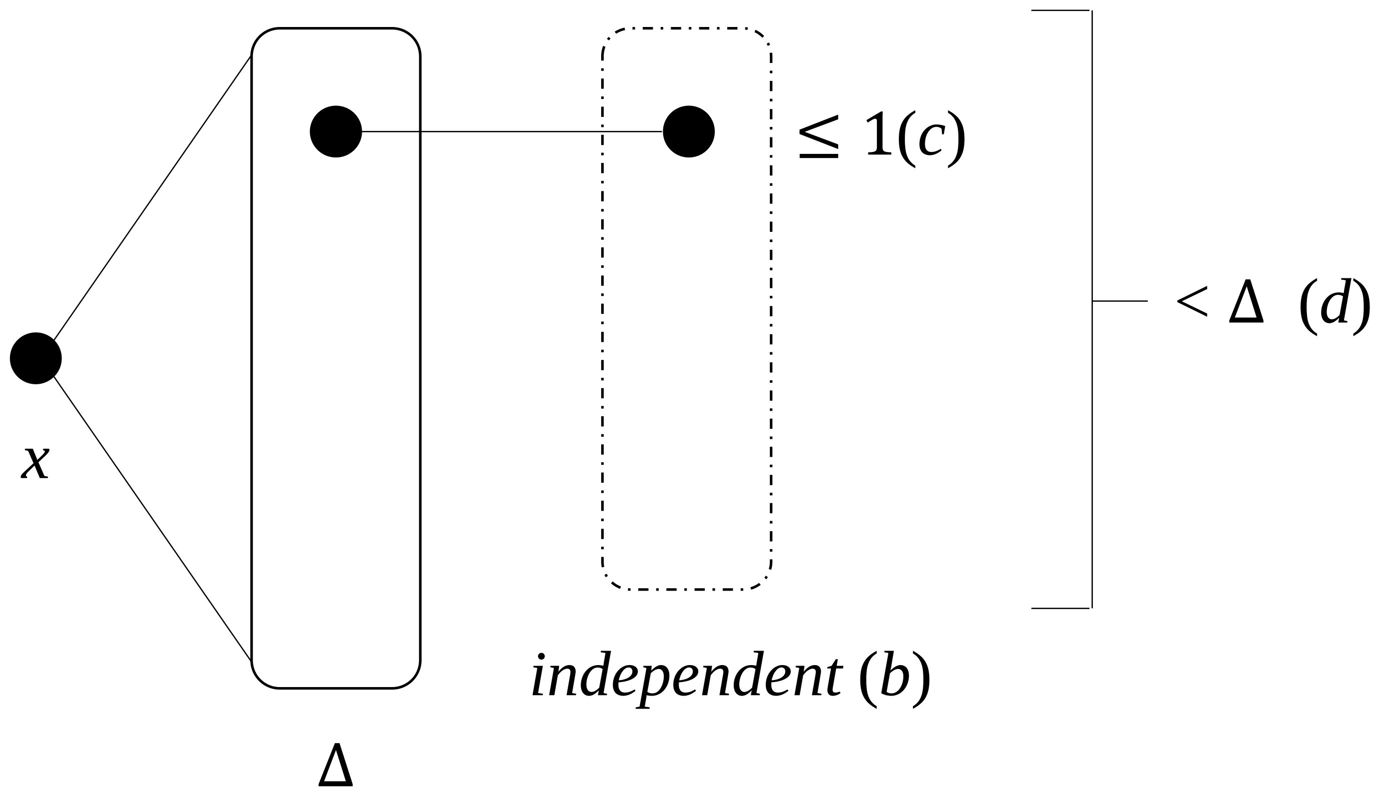
x
Δ



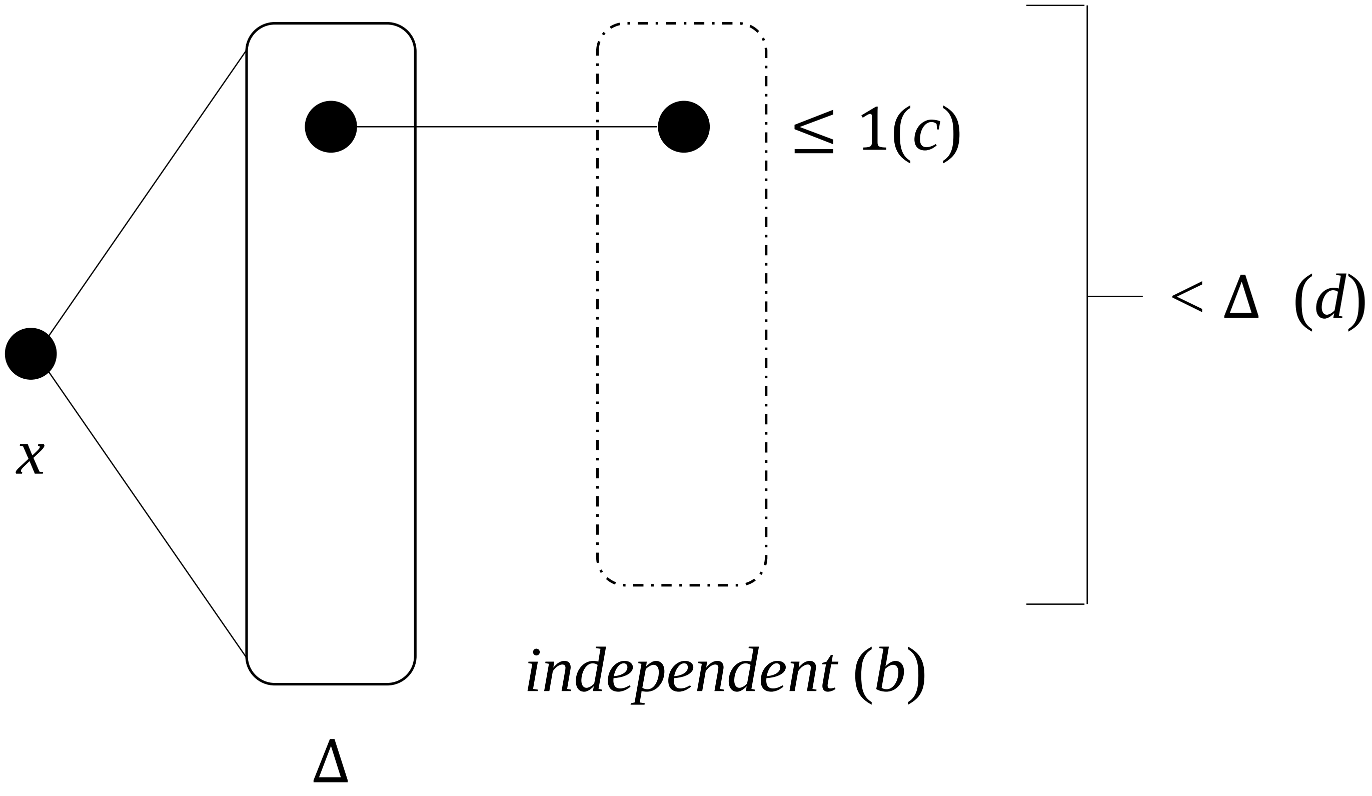
x
Δ
independent (b)



$\leq 1(c)$
x
independent (b)
Δ



$\leq 1(c)$
$< \Delta$ (d)
x
independent (b)
Δ



$$H \in \mathcal{A}_1(\Delta, s)$$

The class $\mathcal{G}_1(\Delta, s)$.

For given positive integers $s > \Delta \geq 2$, a graph G in this class is constructed from a wounded spider $s(p, q)$ where $s = 2p - q + 1$ with the maximum degree vertex x and from $\Delta s - 2s + 2$ distinct graphs from the class $\mathcal{A}_1(\Delta, s)$ by joining vertices between components as follows.

Let $x, a_1, ..., a_{p-q}, b_1, ..., b_{p-q}, c_1, ..., c_q$ be defined by the definition of $s(p, q)$.

(i) For all $1 \leq i \leq p-q$, join a vertex a_i to $\Delta - 2$ components $F_i^1, F_i^2, ..., F_i^{\Delta-2} \in \mathcal{A}_1(\Delta, s)$ at a non-critical vertex $f_i^1, f_i^2, ..., f_i^{\Delta-2}$ respectively. Further, each of f_i^j is not maximum degree vertex of F_i^j for all $1 \leq j \leq \Delta - 2$.

(ii) For all $1 \leq i \leq p-q$, join a vertex b_i to $\Delta-1$ components $R_i^1, R_i^2, ..., R_i^{\Delta-1} \in \mathcal{A}_1(\Delta, s)$ at a non-critical bad vertex $r_i^1, r_i^2, ..., r_i^{\Delta-1}$ respectively. Further, each of r_i^j is not maximum degree vertex of R_i^j for all $1 \leq j \leq \Delta - 1$.

Further, when $q = 1$,

(*iii*) join a vertex c_1 to $\Delta - 1$ components $H_1^1, H_1^2, ..., H_1^{\Delta-1} \in \mathcal{A}_1(\Delta, s)$ at a non-critical bad vertex $h_1^1, h_1^2, ..., h_1^{\Delta-1}$ respectively. Further, each of h_i^j is not maximum degree vertex of H_i^j for all $1 \leq j \leq \Delta - 1$.

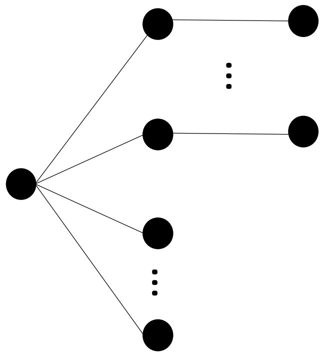
When $q \geq 2$, for all $1 \leq i \leq q$, join a vertex c_i to $\Delta - 1$ components $H_i^1, H_i^2, ..., H_i^{\Delta-1} \in \mathcal{A}_1(\Delta, s)$ at a vertex $h_i^1, h_i^2, ..., h_i^{\Delta-1}$ respectively, in such a way that :

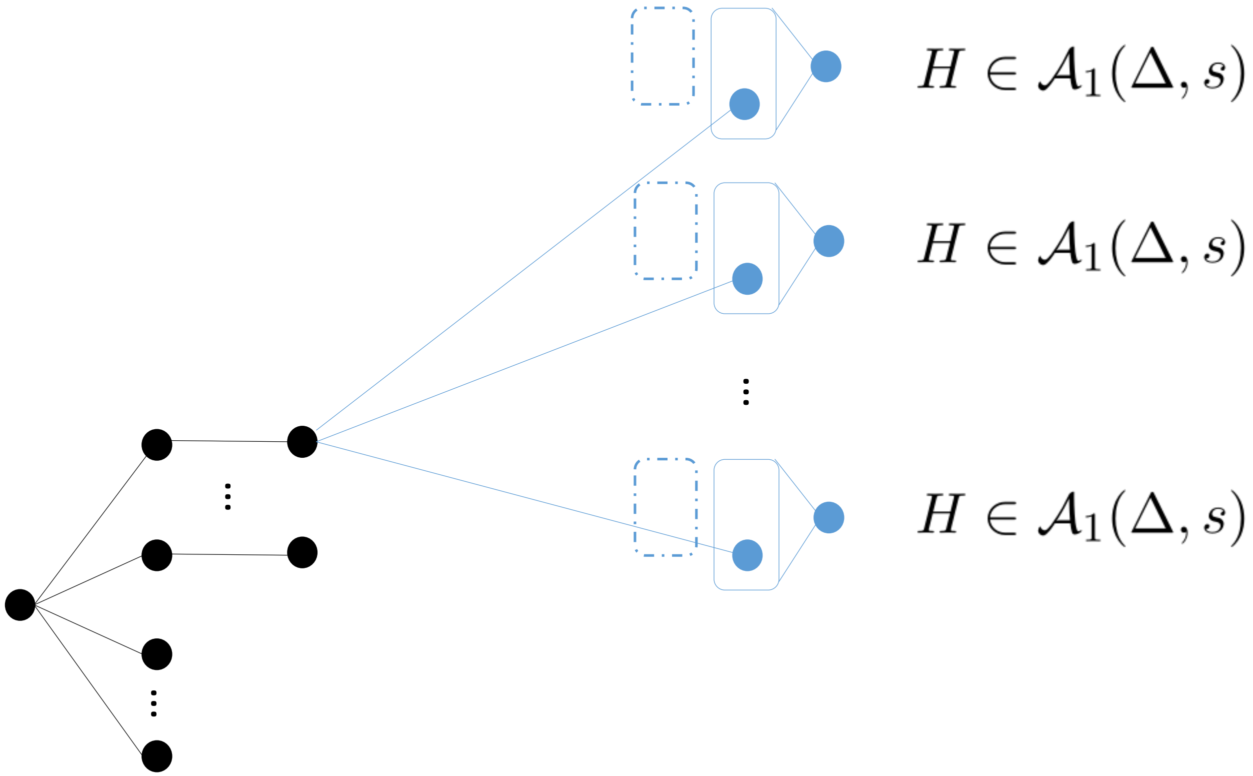
(*iv*) Each of h_i^j is not maximum degree vertex of H_i^j for all $1 \leq j \leq \Delta - 1$.

(*v*) At most one of these $\Delta - 1$ vertices $h_i^1, h_i^2, ..., h_i^{\Delta-1}$ is critical.

(*vi*) There exists $1 \leq i' \leq q$ such that $c_{i'}$ is adjacent to all non-critical vertices $h_{i'}^1, h_{i'}^2, ..., h_{i'}^{\Delta-1}$.

(*vii*) There exists $1 \leq i'' \leq q$ such that $c_{i''}$ is adjacent to all bad vertices $h_{i''}^1, h_{i''}^2, ..., h_{i''}^{\Delta-1}$.





$H \in \mathcal{A}_1(\Delta, s)$
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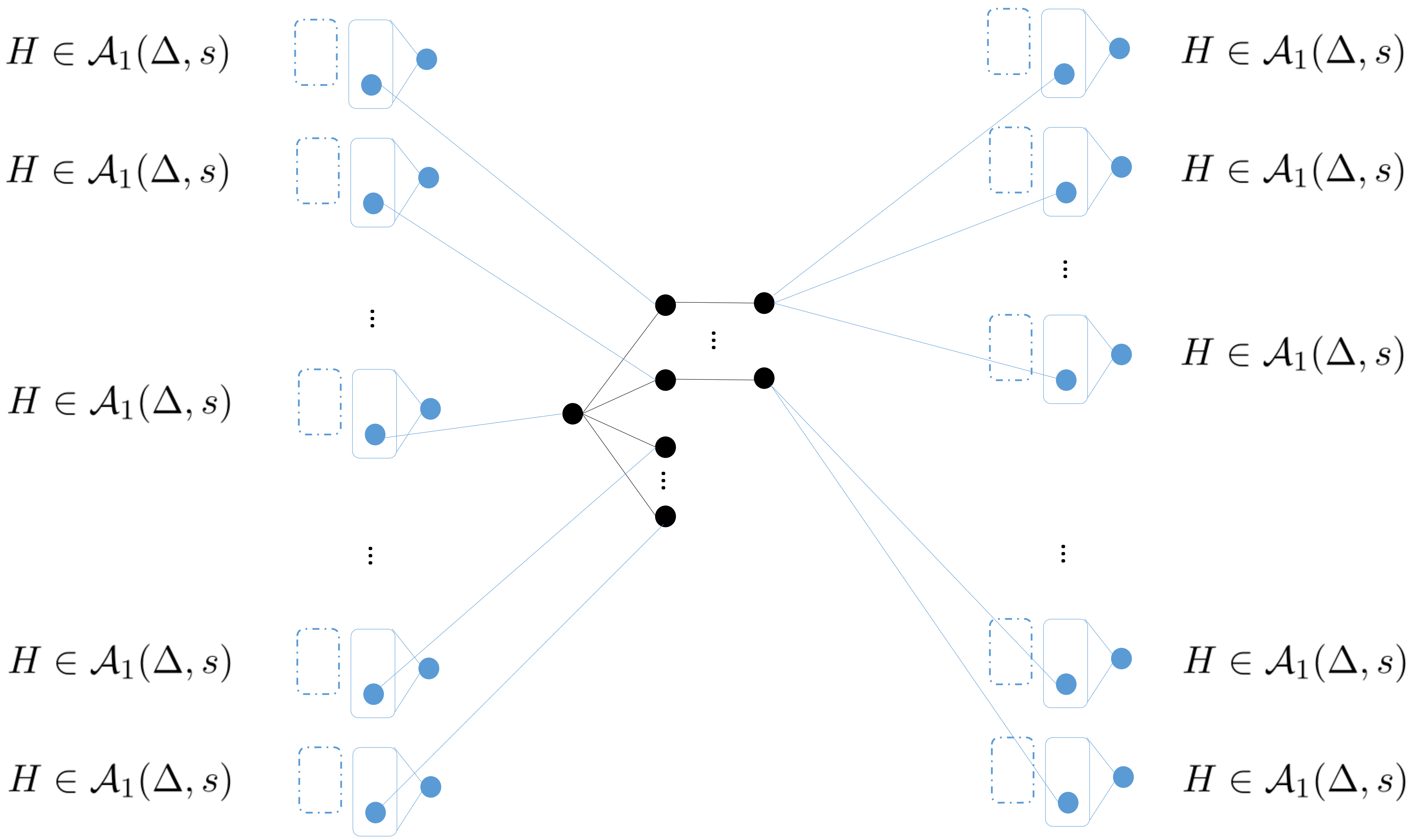
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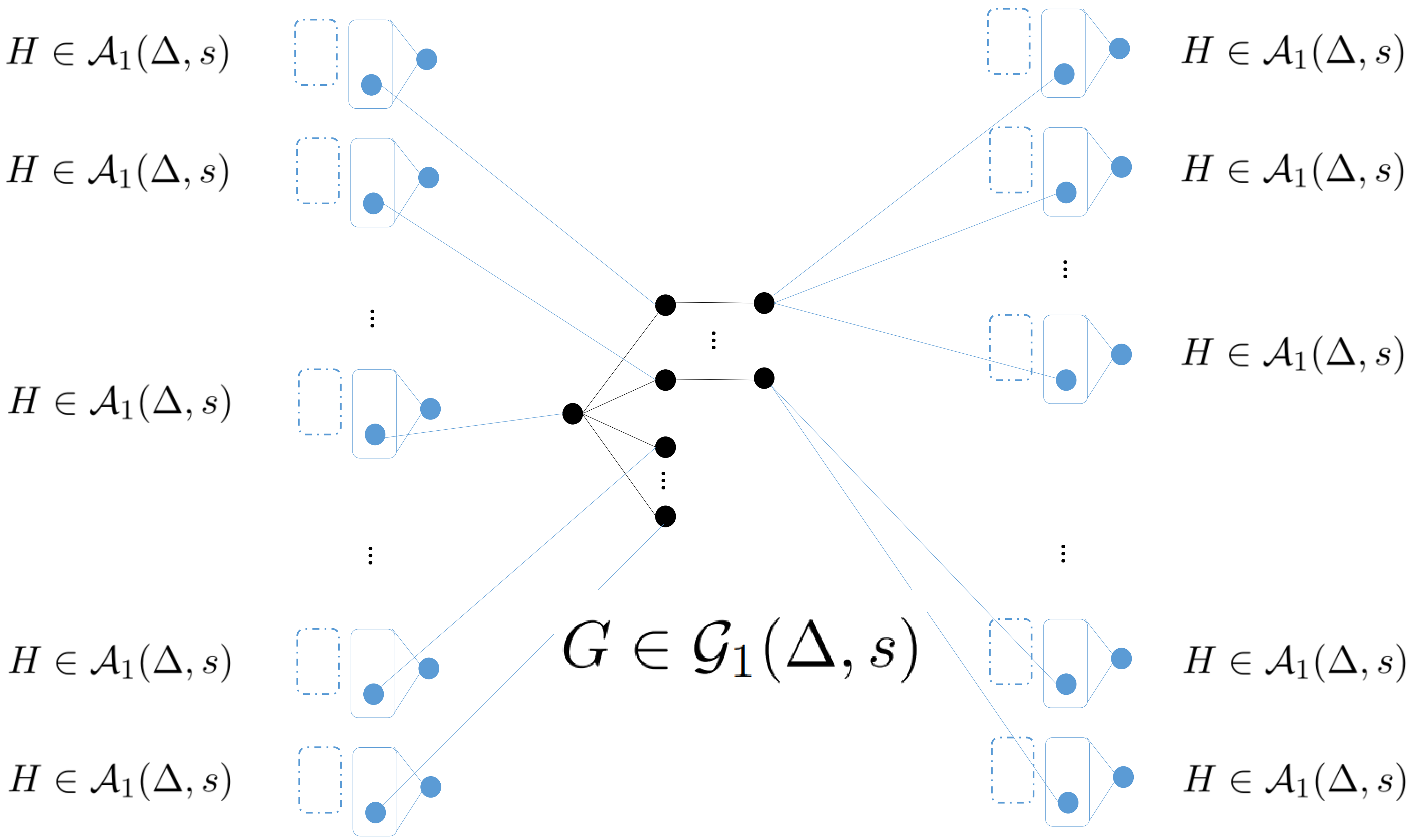
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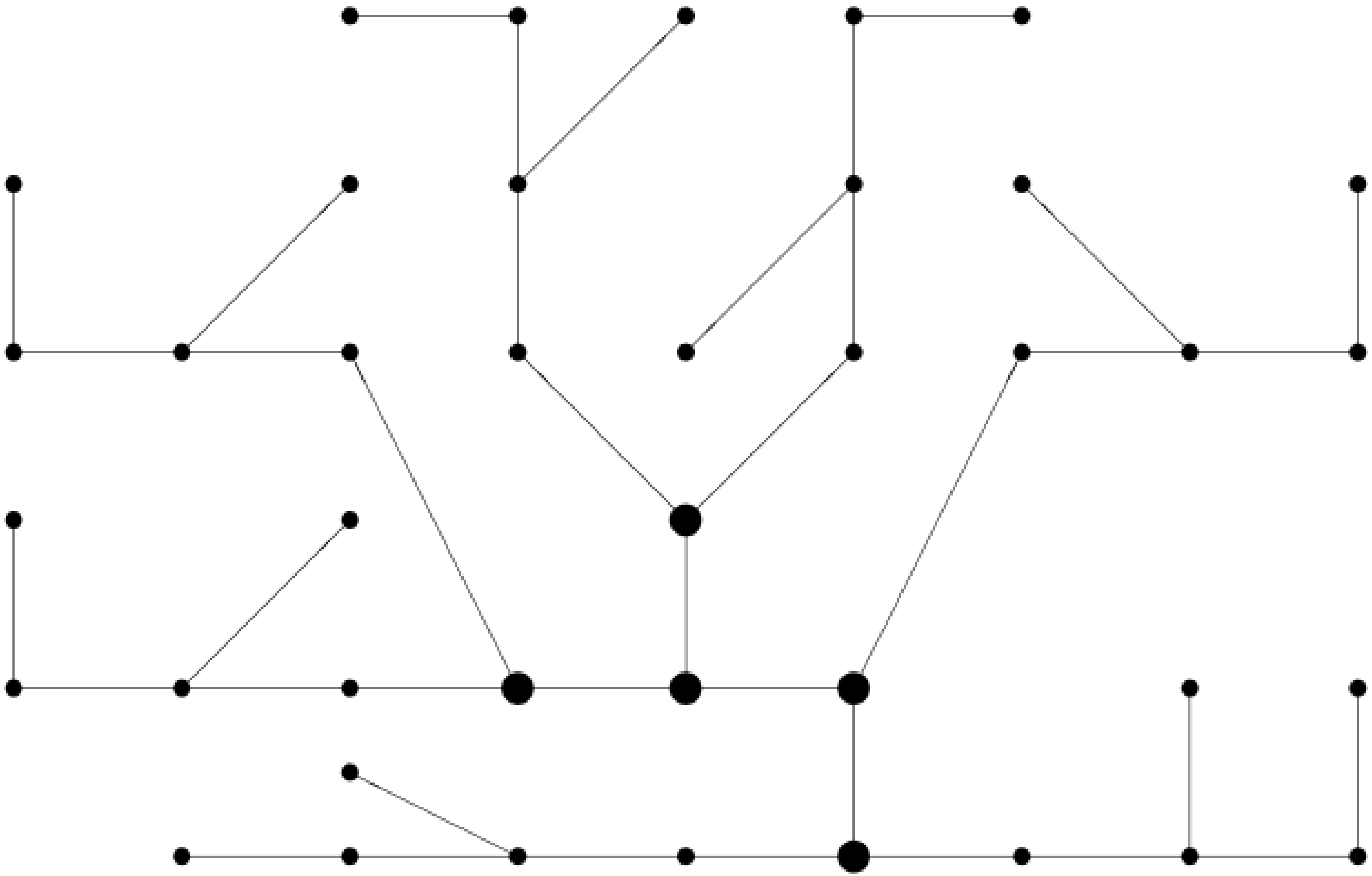
$H \in \mathcal{A}_1(\Delta, s)$
$G \in \mathcal{G}_1(\Delta, s)$
$H \in \mathcal{A}_1(\Delta, s)$
$H \in \mathcal{A}_1(\Delta, s)$
$H \in \mathcal{A}_1(\Delta, s)$
$H \in \mathcal{A}_1(\Delta, s)$
$H \in \mathcal{A}_1(\Delta, s)$

The graph G satisfies the lower bound of Theorem 1

if and only if

$$G \in \mathcal{G}_1(\Delta, s)$$



A graph $G \in \mathcal{G}_1(\Delta, s)$ *when* Δ= 3 *and s*=5

By more or less similar arguments, we prove that

Theorem 2 *Let G be a connected graph. Then*

$$g(\Delta) \leq s_c(G) \leq \lceil \frac{\gamma_c(G)(\Delta-1)+2}{2} \rceil$$

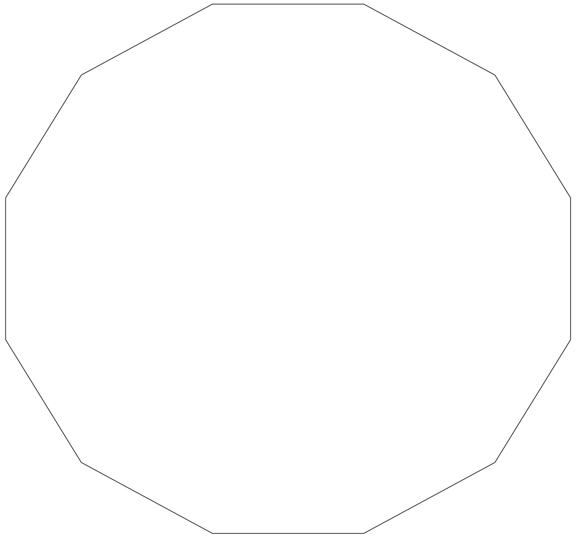
where $g(\Delta) = \frac{\gamma_c(G)+2}{3}$ *when* $\Delta = 2$ *and* $g(\Delta) = \frac{\Delta-5+\sqrt{\Delta^2-2\Delta+4(\Delta-2)\gamma_c(G)+9}}{2(\Delta-2)}$ *when* $\Delta \geq 3$.

The upper bounds of both theorems are shown to be sharp. Further, we characterize all graphs satisfying the lower bound of each theorem.

Upper bound

Some extremal graphs satisfying the equality $s_c(G) = \lceil (\gamma_c(G)(\Delta - 1) + 2)/2 \rceil$ *are*

When $\Delta = 2$



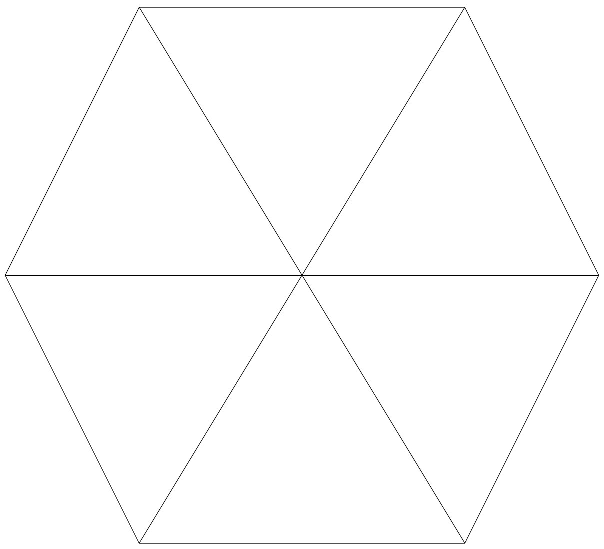
C_n

Upper bound

Some extremal graphs satisfying the equality $s_c(G) = \lceil(\gamma_c(G)(\Delta - 1) + 2)/2\rceil$ *are*

When $\gamma_c(G) = 2$, n = 2Δ and Δ is odd

$$C_n\langle 1,3,\ldots,\Delta\rangle$$



Upper bound

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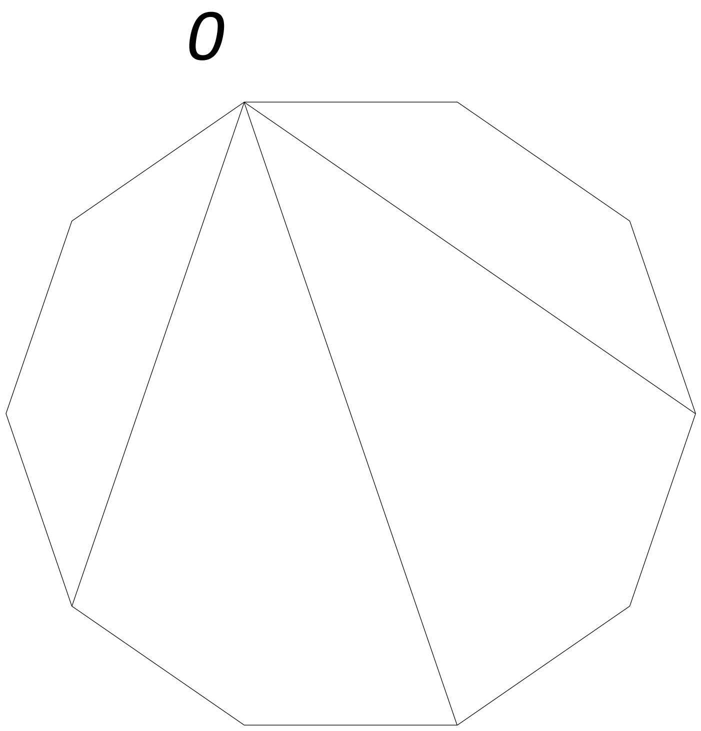
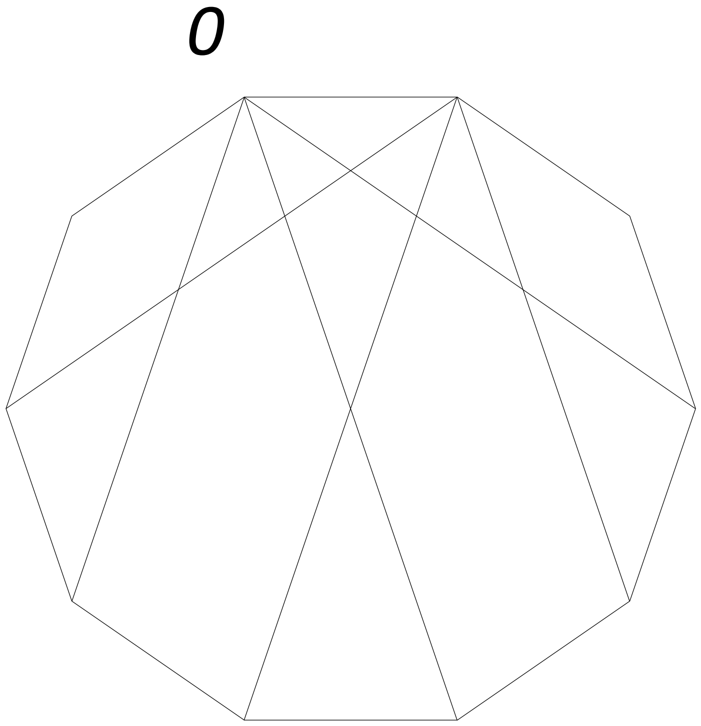


Upper bound

Some extremal graphs satisfying the equality $s_c(G) = \lceil(\gamma_c(G)(\Delta - 1) + 2)/2\rceil$ *are*

When $\boldsymbol{\gamma_c(G) = 2}$, n = 2Δ and Δ is odd

$C_n\langle 1,3,\ldots,\Delta\rangle$

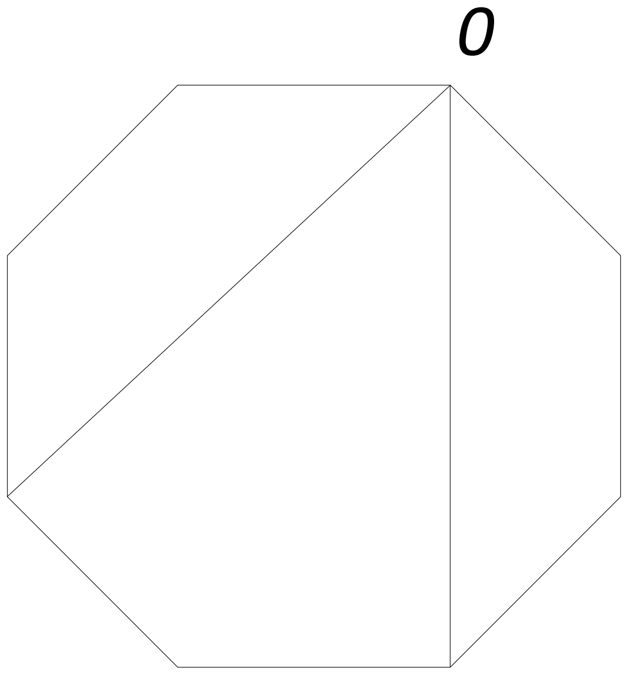


Upper bound

Some extremal graphs satisfying the equality $s_c(G) = \lceil(\gamma_c(G)(\Delta - 1) + 2)/2\rceil$ *are*

When $\boldsymbol{\gamma_c(G) = 2}$ and n = 2Δ and Δ is even

$C_n\langle 1,3, \dots, \Delta - 1\rangle$

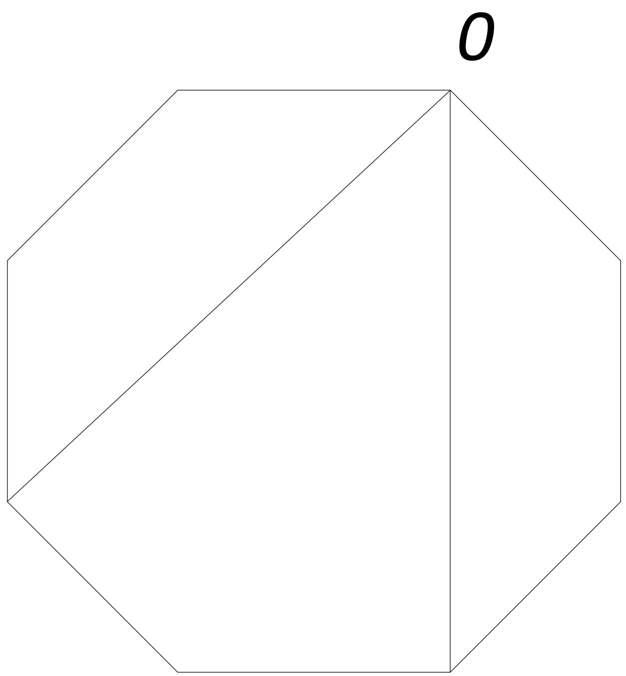


Upper bound

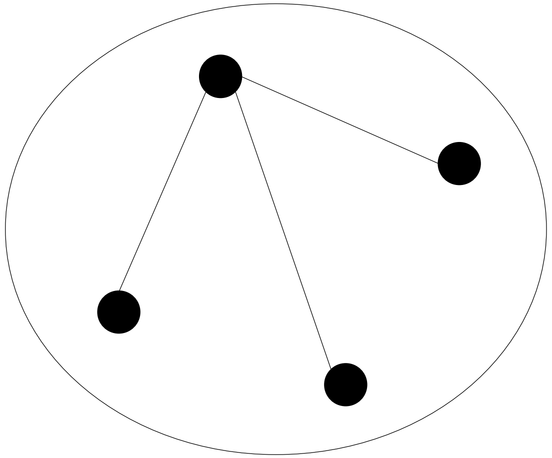
Some extremal graphs satisfying the equality $s_c(G) = \lceil(\gamma_c(G)(\Delta - 1) + 2)/2\rceil$ *are*

When $\gamma_c(G) = 2$ and n = 2Δ and Δ is even

$C_n\langle 1,3, \ldots , \Delta - 1\rangle$

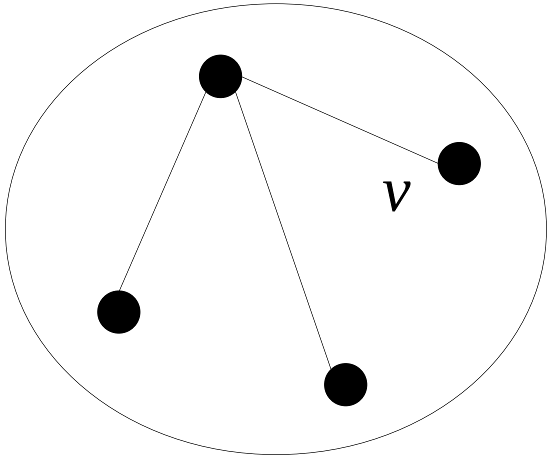


Lower bound



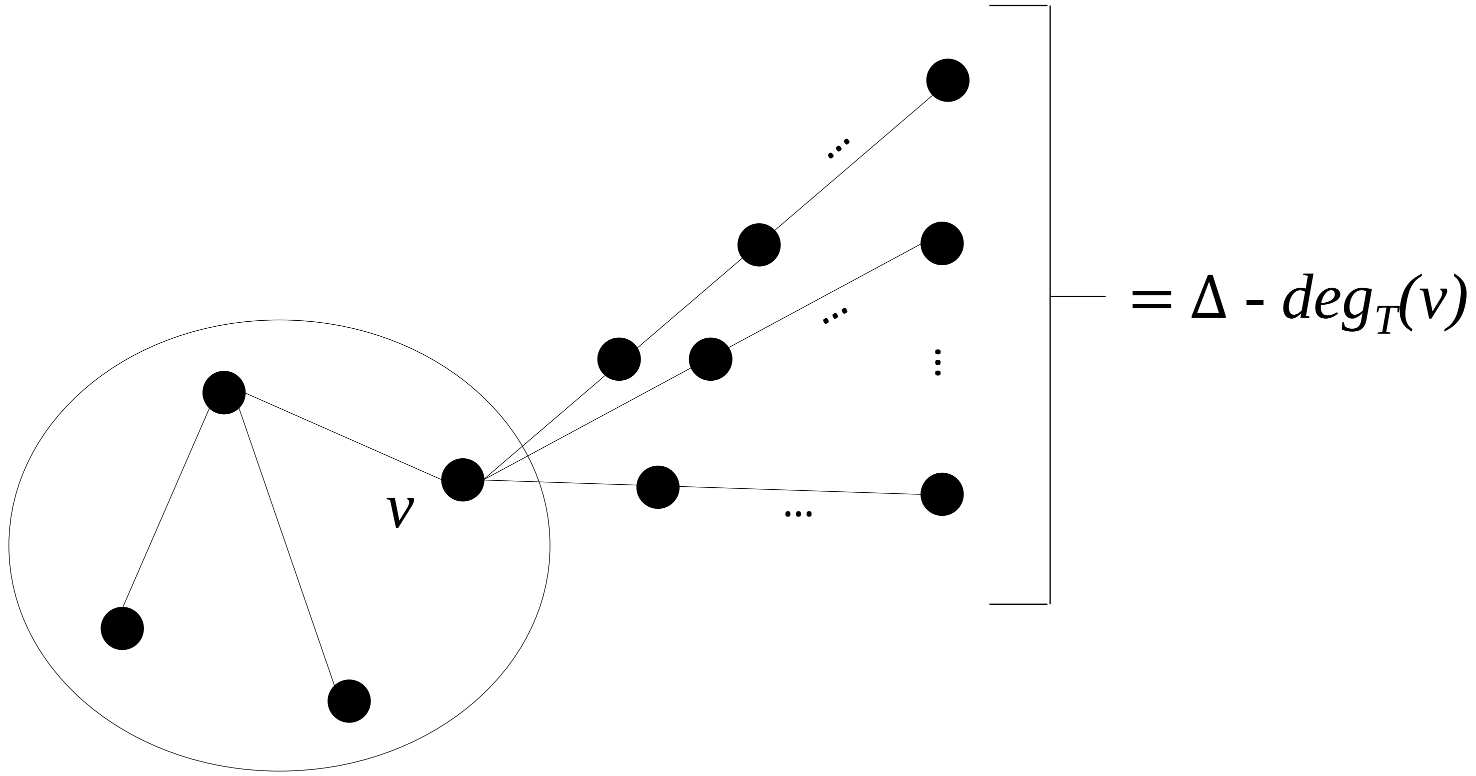
A tree T of order s_c with maximum degree Δ

Lower bound



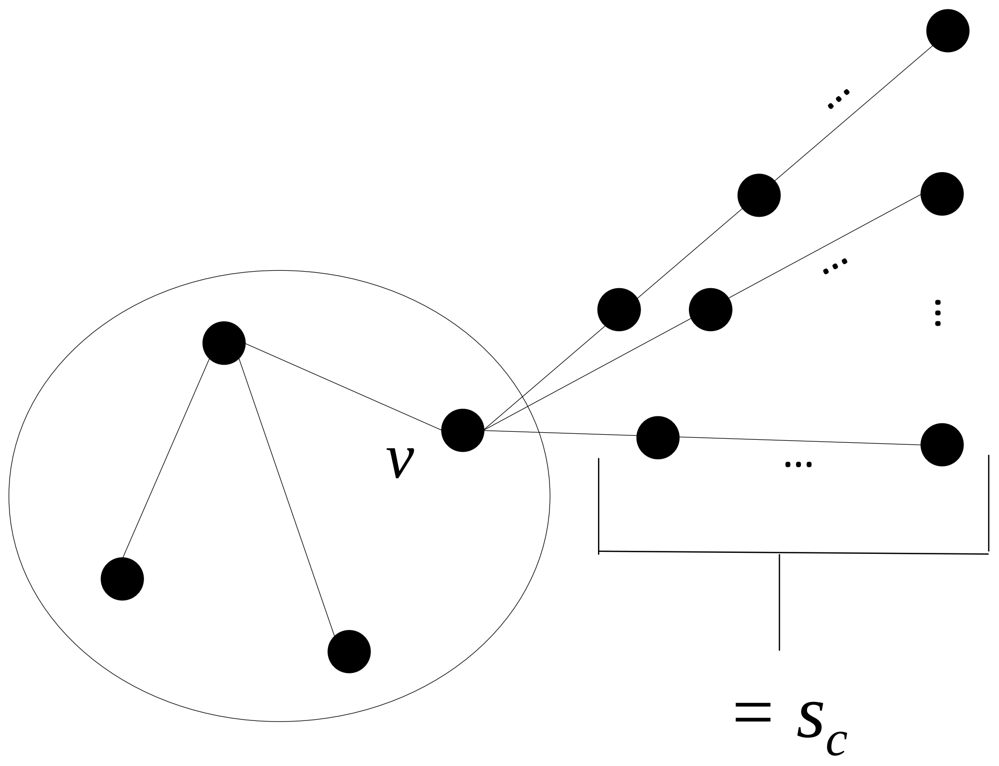
A tree T of order s_c with maximum degree Δ

Lower bound



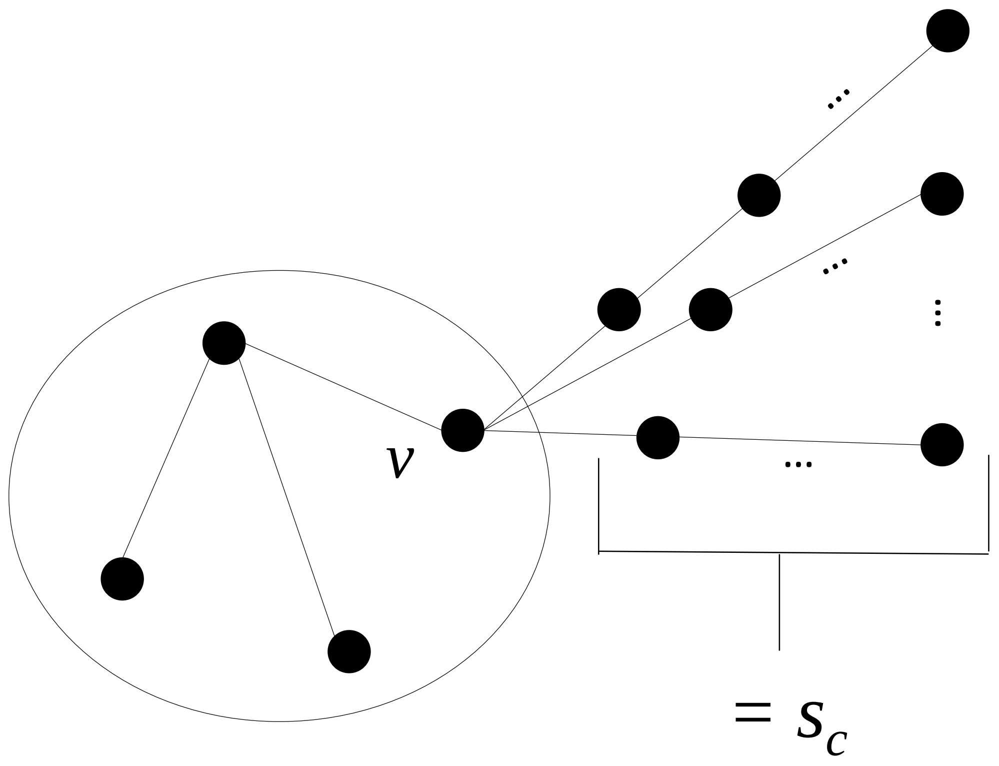
A tree T of order s_c with maximum degree Δ

Lower bound



A tree T of order s_c with maximum degree Δ

Lower bound



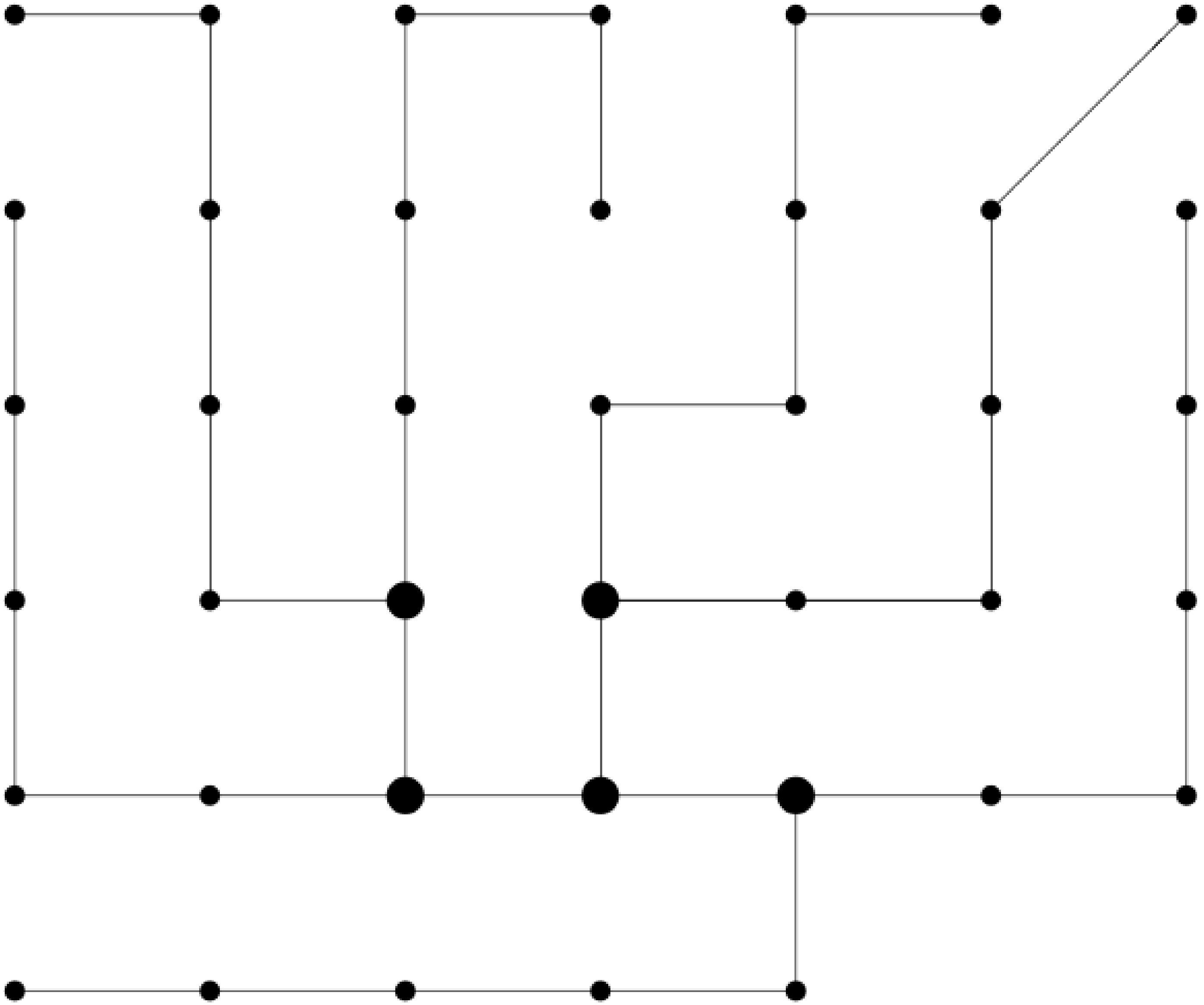
$$G \in \mathcal{G}_2(\Delta, s)$$

Lower bound

A graph G satisfies the lower bound of Theorem 2

if and only if

$$G \in \mathcal{G}_2(\Delta, s)$$



A graph $G \in \mathcal{G}_2(\Delta, s)$ *when* $\Delta = 3$ *and* $s=5$

Thank you