Designs in the generalised symmetric group

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Paderborn University

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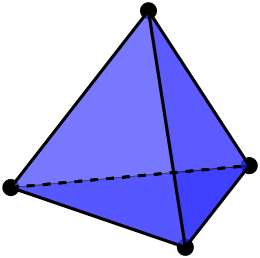
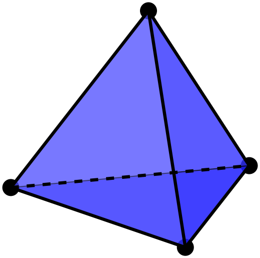
Joint work with Kai-Uwe Schmidt

Introduction





symmetry group S_4

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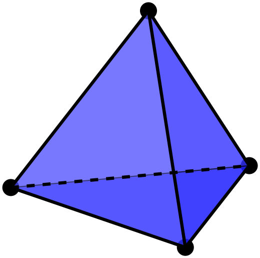
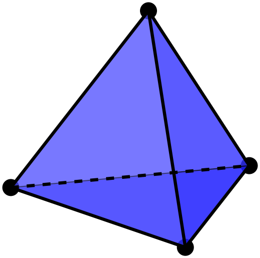


symmetry group S_4

real space:

Introduction



symmetry group S_4

real space:

- tetrahedron (simplex)

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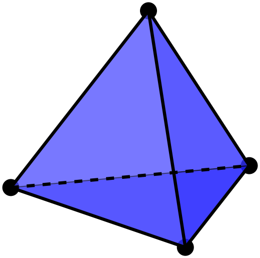

symmetry group S_4

real space:

- tetrahedron (simplex)
- cube (hypercube)

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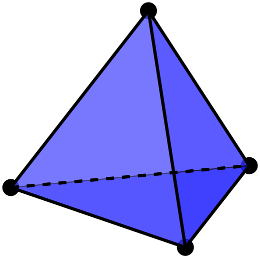

symmetry group S_4

real space:

- tetrahedron (simplex)
- cube (hypercube)
- octahedron (hyperoctahedron)

Regular polytopes

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made up of vertices, edges, faces, cells, . . .

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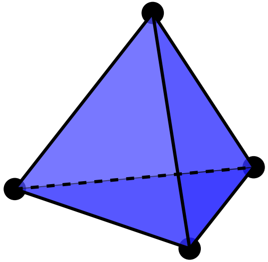

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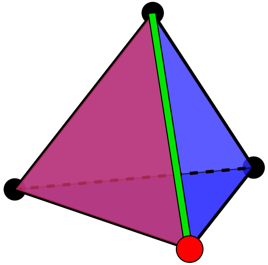

Transitivity in polytopes

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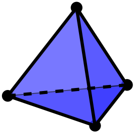

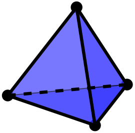

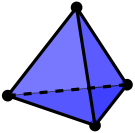

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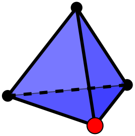

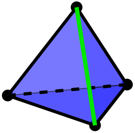

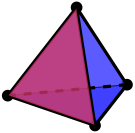

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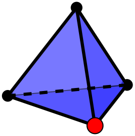

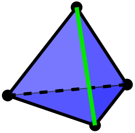

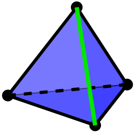

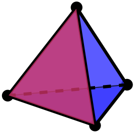

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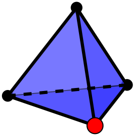

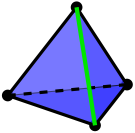

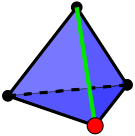

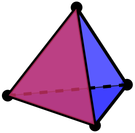

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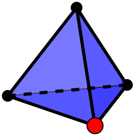

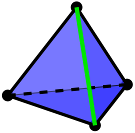

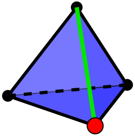

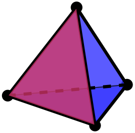

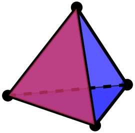

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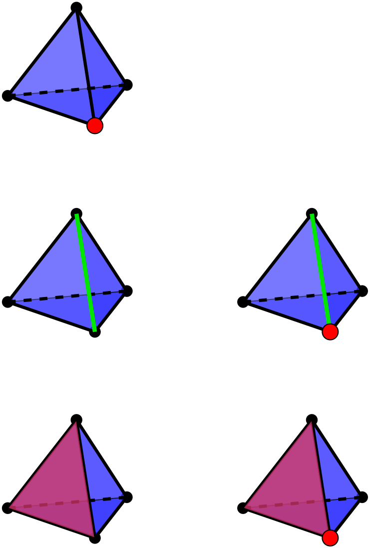

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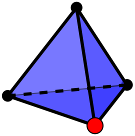

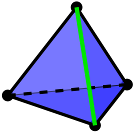

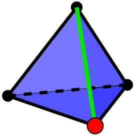

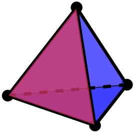

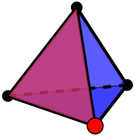

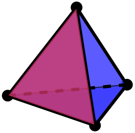

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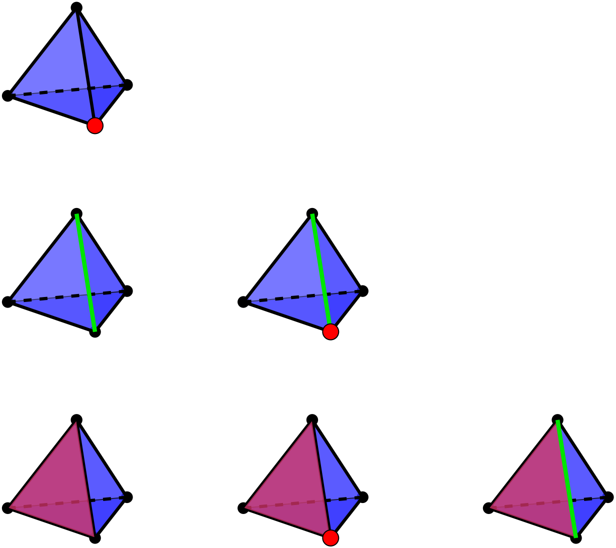

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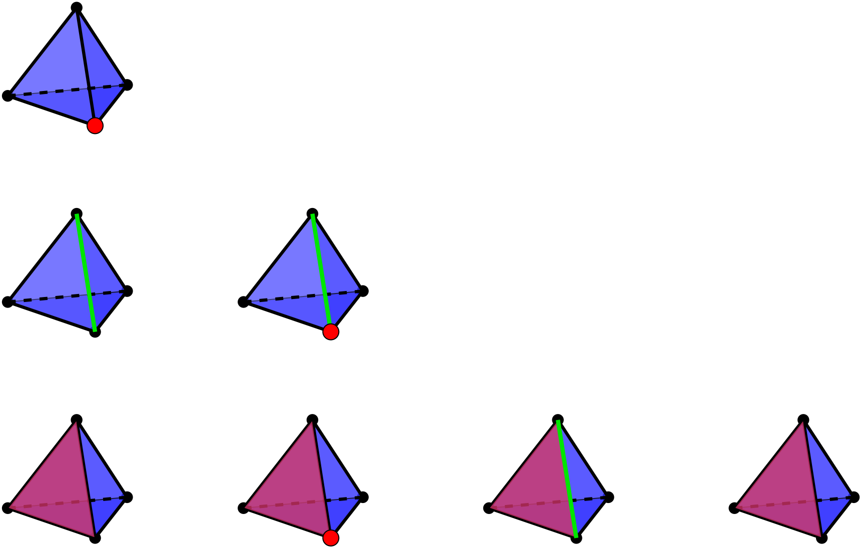

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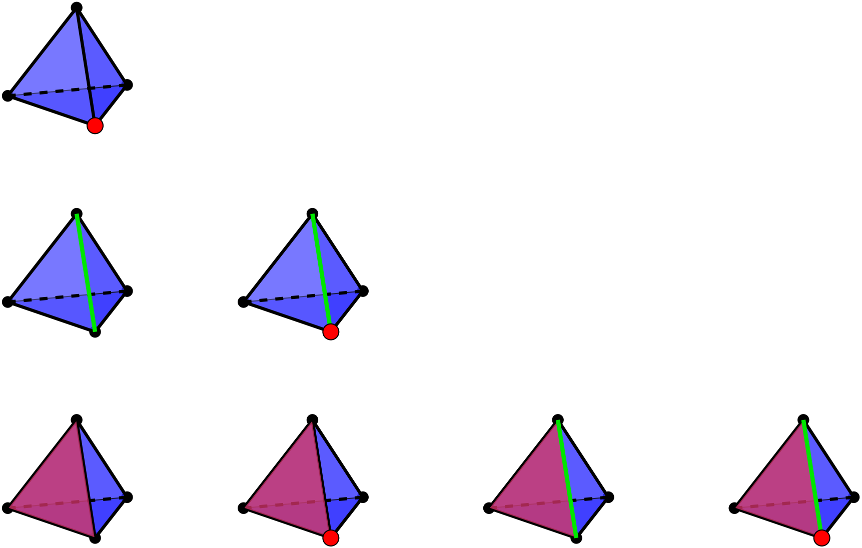

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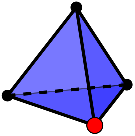

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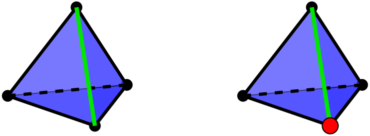

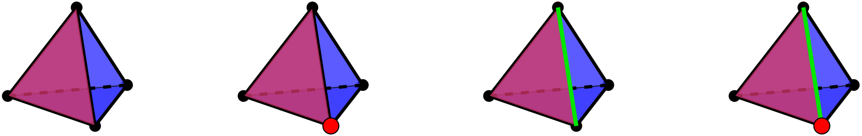

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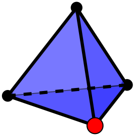

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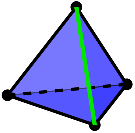

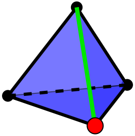

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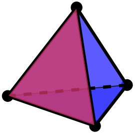

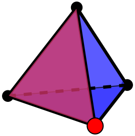

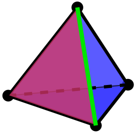

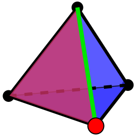

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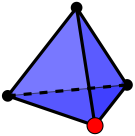

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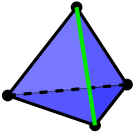

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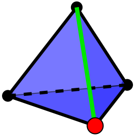

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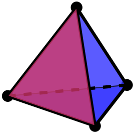

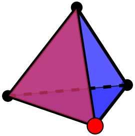

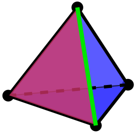

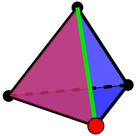

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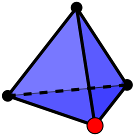

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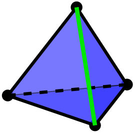

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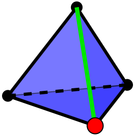

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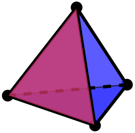

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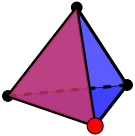

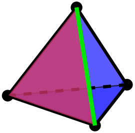

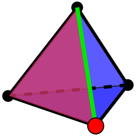

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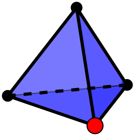

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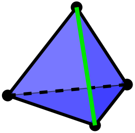

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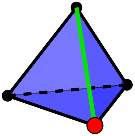

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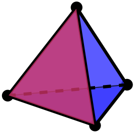

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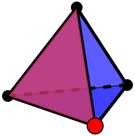

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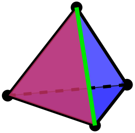

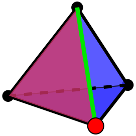

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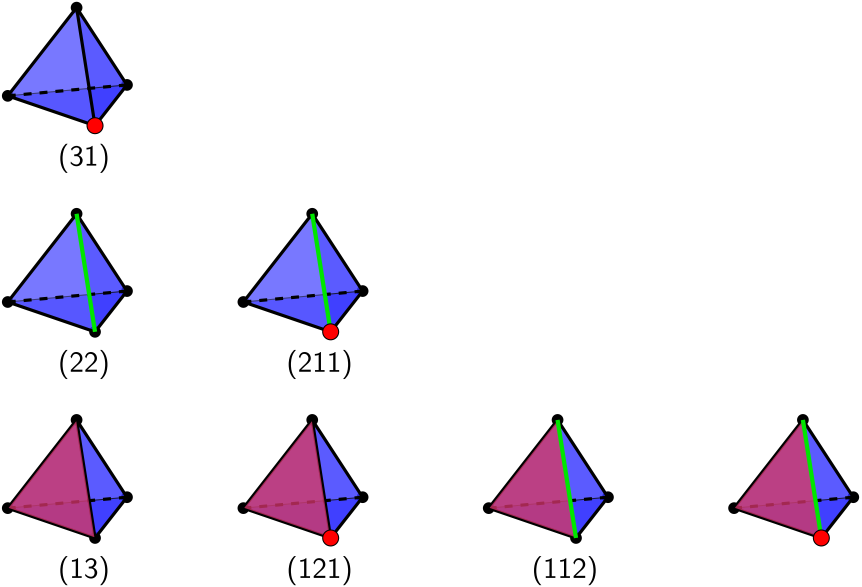

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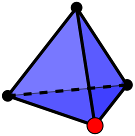

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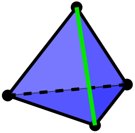

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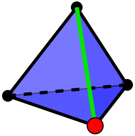

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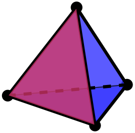

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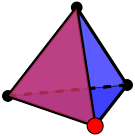

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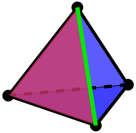
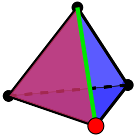
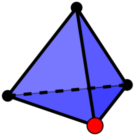


(112)



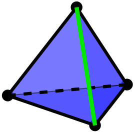
(1111)

Transitivity in polytopes



(31)

λ-transitive set of symmetries $\longleftrightarrow$ design in S_n



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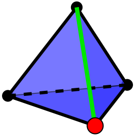

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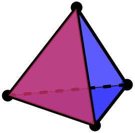

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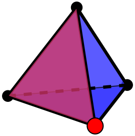

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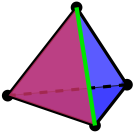

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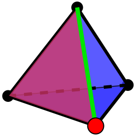

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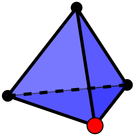

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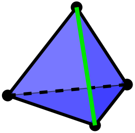

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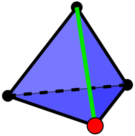

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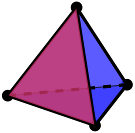

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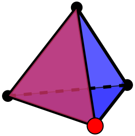

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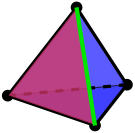

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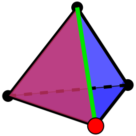

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Transitive sets

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  $\rightarrow$ unifiying framework to study codes and designs

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This motivates the present general definition [of T-designs], the 'conjecture' being that T-designs will often have interesting properties.

– Delsarte's thesis, 1973

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Conjugacy class scheme of a finite group G:

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Conjugacy class scheme of S_n:

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Theorem [Martin, Sagan 2006]

Let λ be a partition of n and $D \subseteq S_n$. Then:
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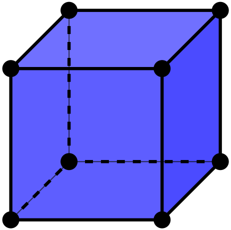

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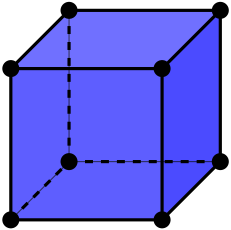

$C_2 \wr S_n$

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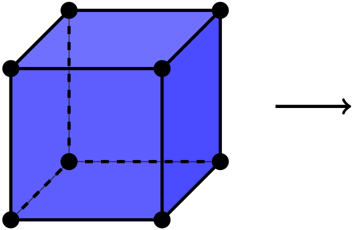

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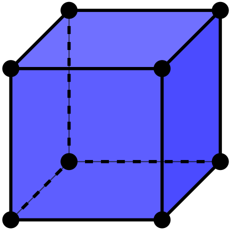

$\longrightarrow$

?

(complex regular polytope)

$C_2 \wr S_n$ $\qquad$ $C_r \wr S_n$

Transitivity in $C_r \wr S_n$

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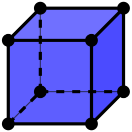

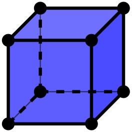

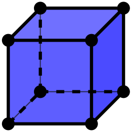

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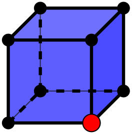

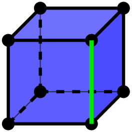

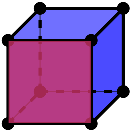

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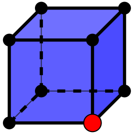

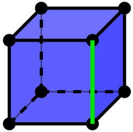

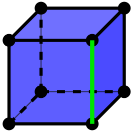

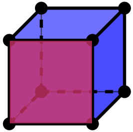

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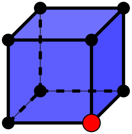

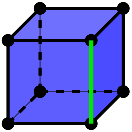

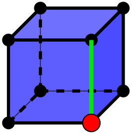

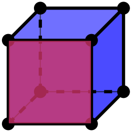

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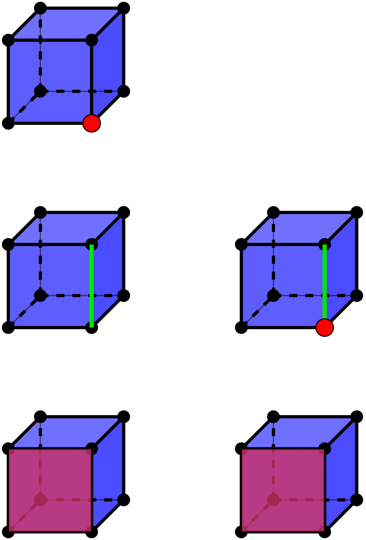

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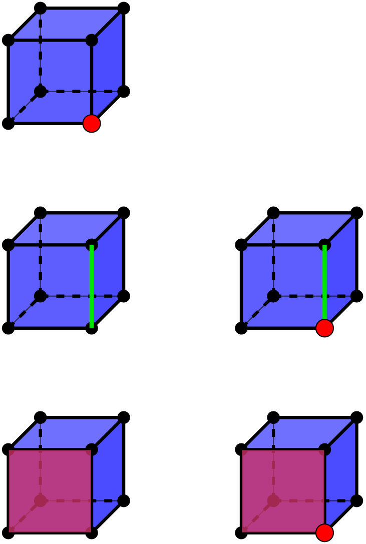

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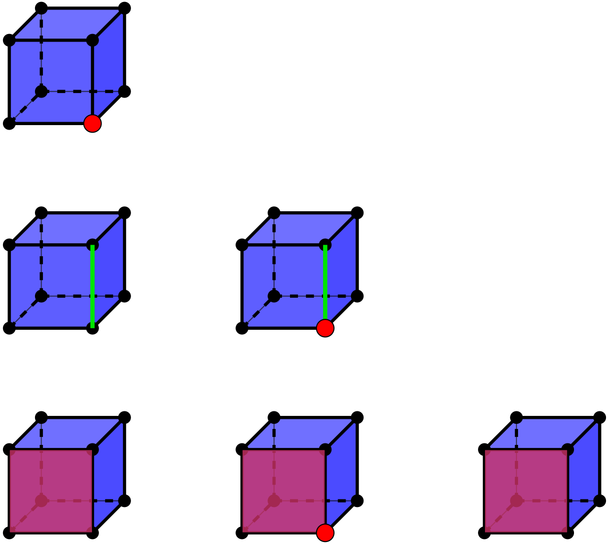

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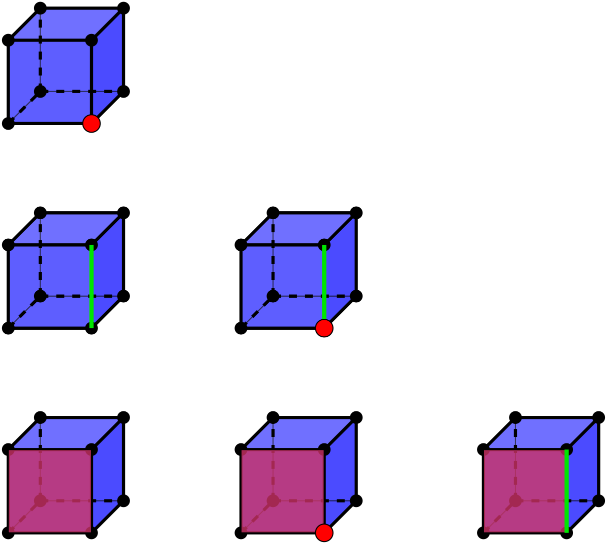

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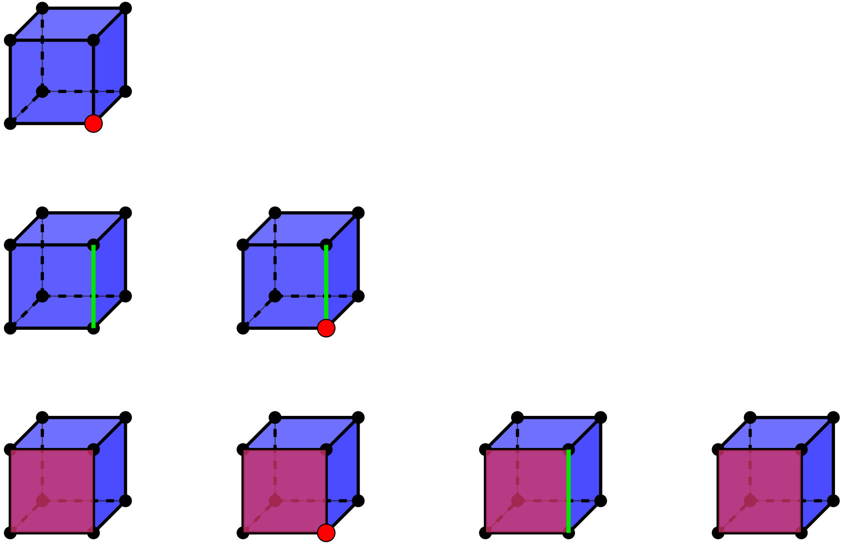

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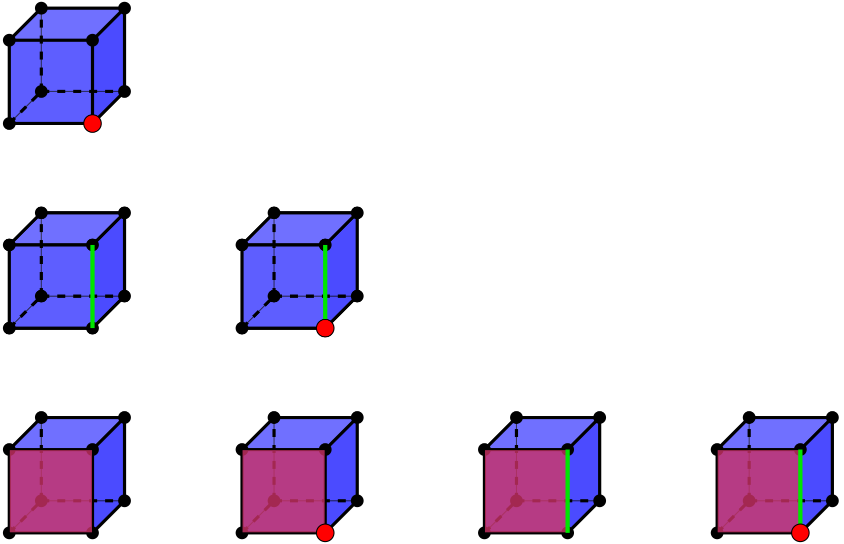

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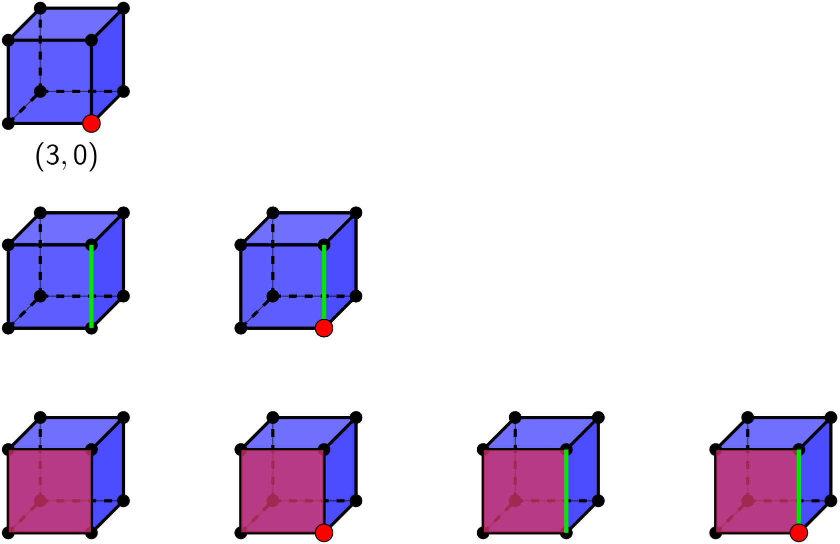

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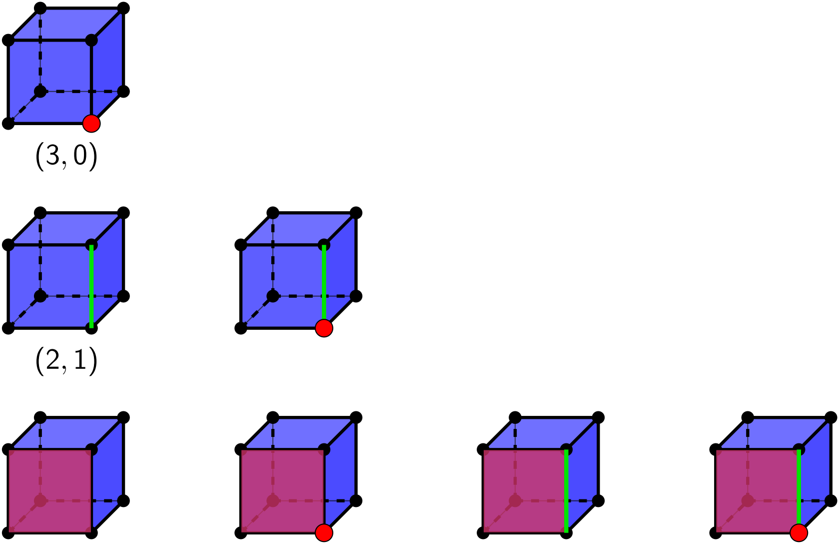

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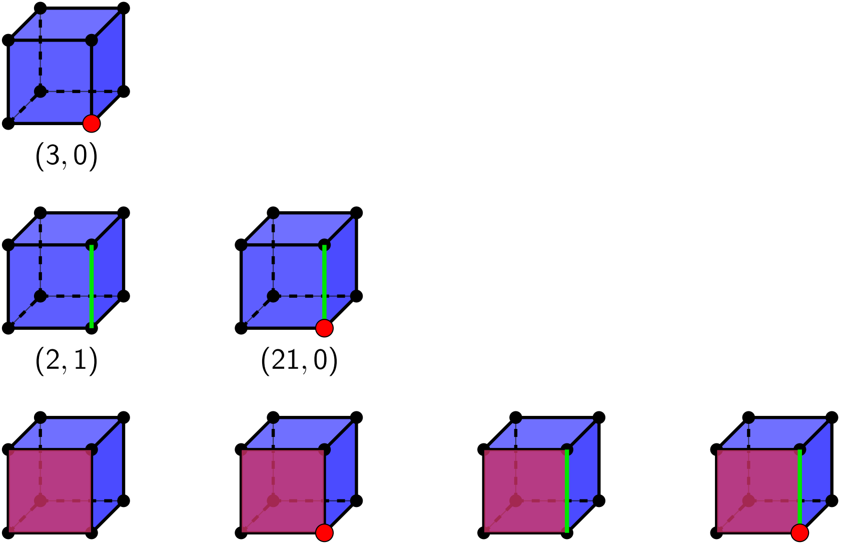

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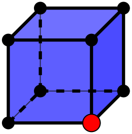

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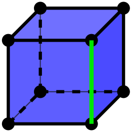

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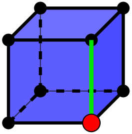

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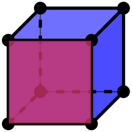

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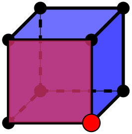

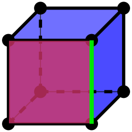

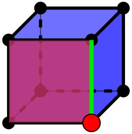

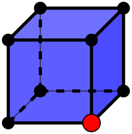

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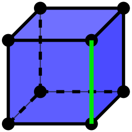

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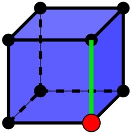

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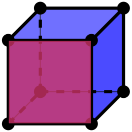

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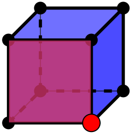

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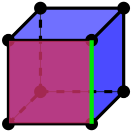

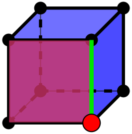

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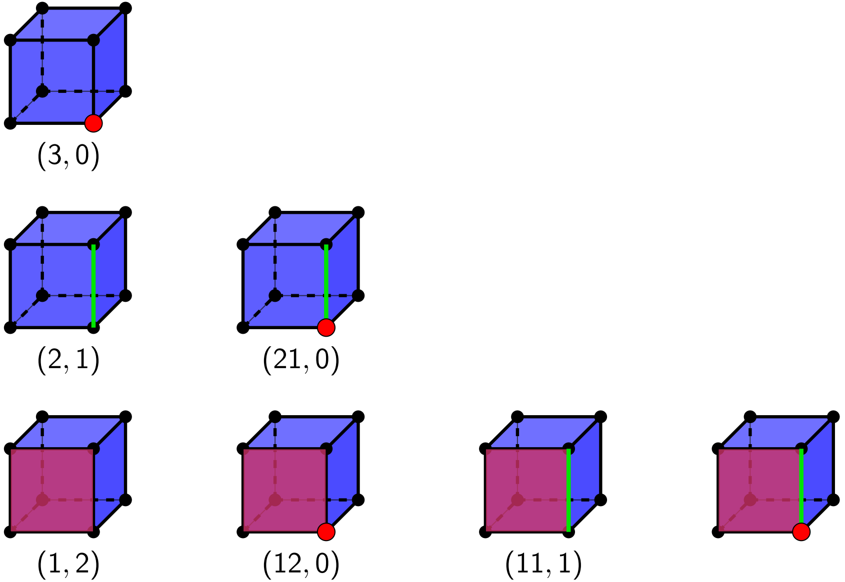

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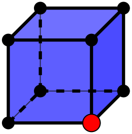

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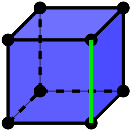

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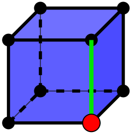

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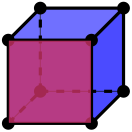

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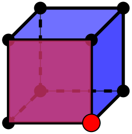

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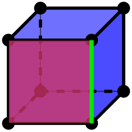

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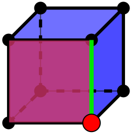

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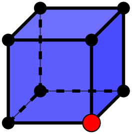

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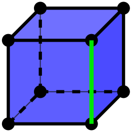
(λ, k)-transitive set of symmetries $\longleftrightarrow$ design in $C_r \wr S_n$



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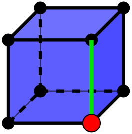

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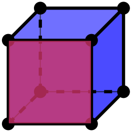

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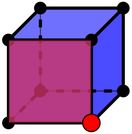

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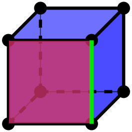

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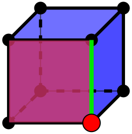

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Characterisation of designs in $C_r \wr S_n$

Theorem [Martin, Sagan 2006]

Let λ be a partition of n and $D \subseteq S_n$. Then:
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Theorem also works for more general transitivity types

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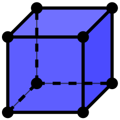

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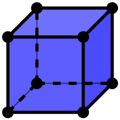

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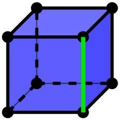

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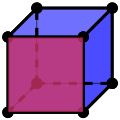

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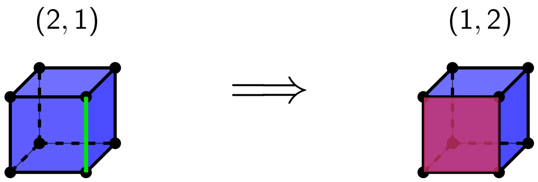

Construction

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Interpret $C_r \wr S_n$ as coloured permutations:

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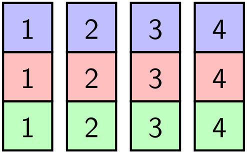

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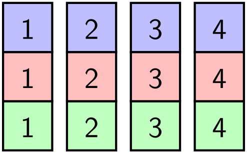

λ-transitive set in S_n

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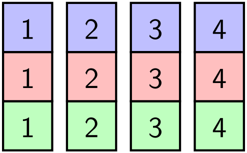

λ-transitive set in S_n + orthogonal array

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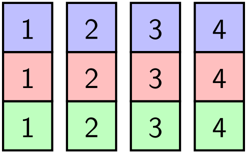

λ-transitive set in S_n + orthogonal array = $\underline{\lambda}$-transitive set in $C_r \wr S_n$

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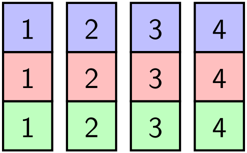

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- S_n: recursive construction by Martin and Sagan

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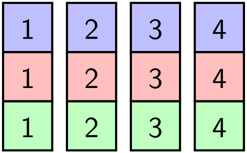

λ-transitive set in S_n + orthogonal array = $\underline{\lambda}$-transitive set in $C_r \wr S_n$

- S_n: recursive construction by Martin and Sagan
- orthogonal arrays: existence by Kuperberg, Lovett and Peled

Thank you for your attention!

The relation $\to$

In $C_3 \wr S_{10}$ we have $(4211, 2) \to (3, 211, 3)$ because

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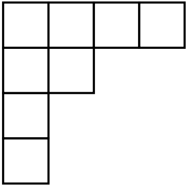
(4211)

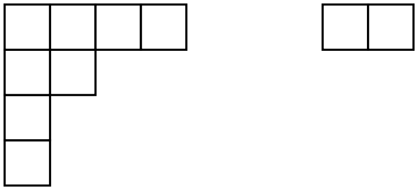


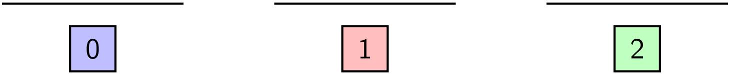
(2)

The relation $\rightarrow$

In $C_3 \wr S_{10}$ we have $(4211, 2) \rightarrow (3, 211, 3)$ because

(4211) (2)





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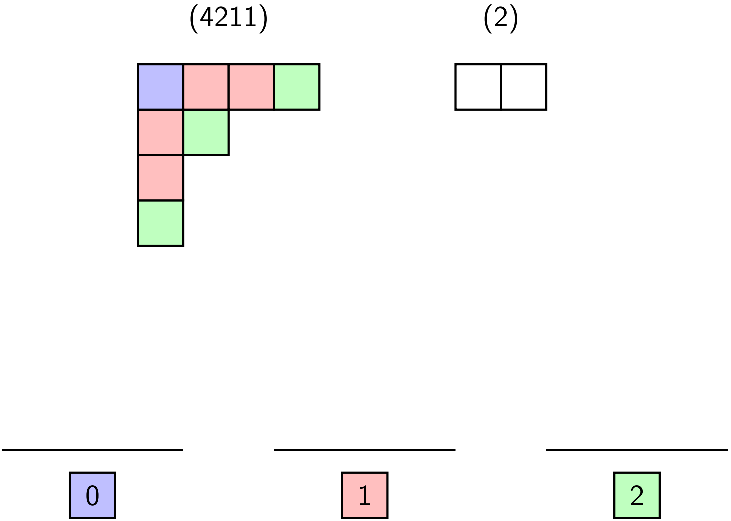

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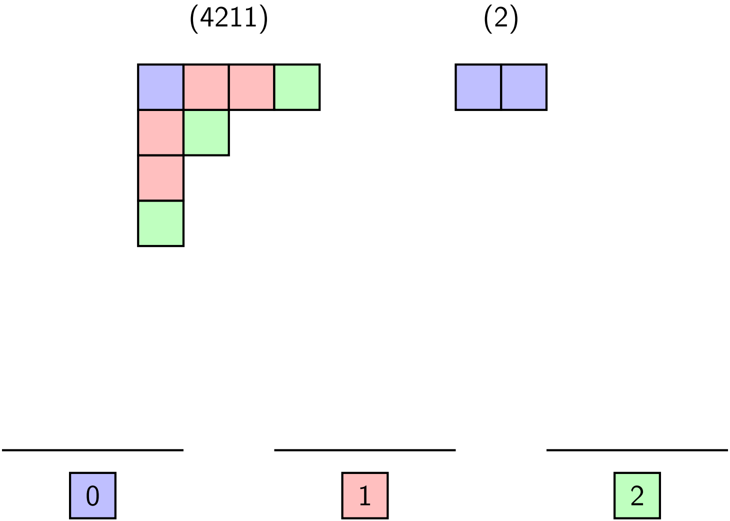

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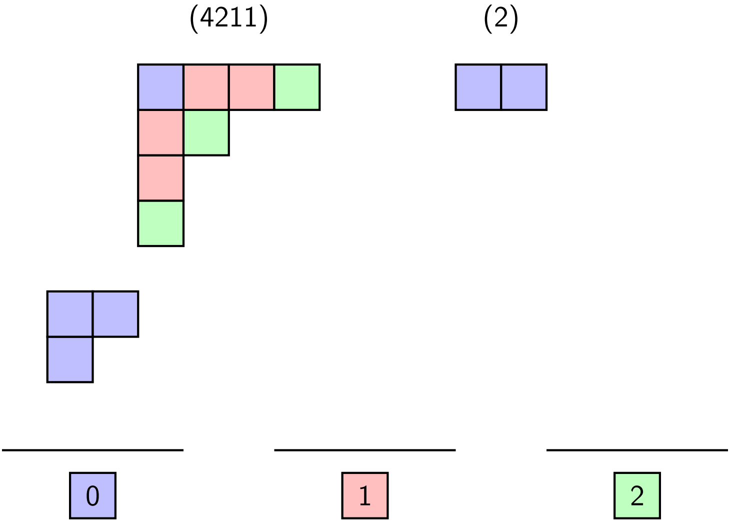

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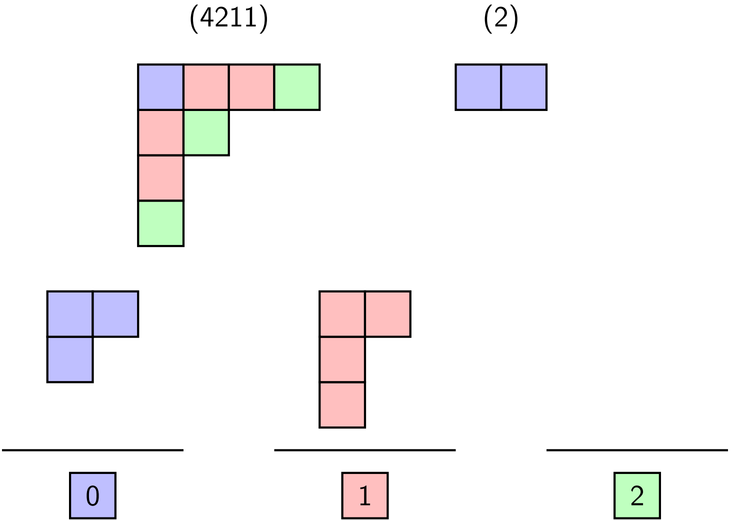

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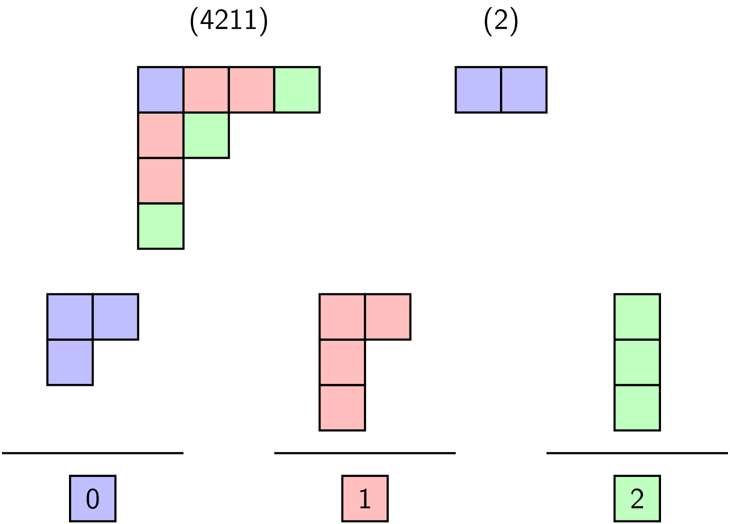

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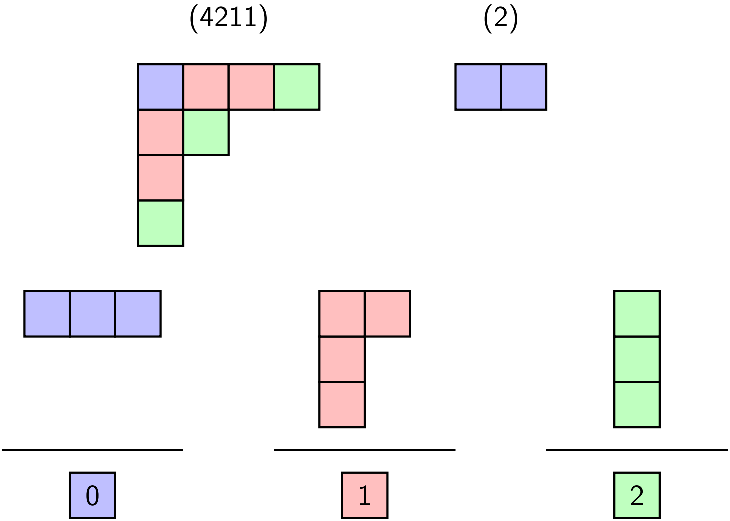

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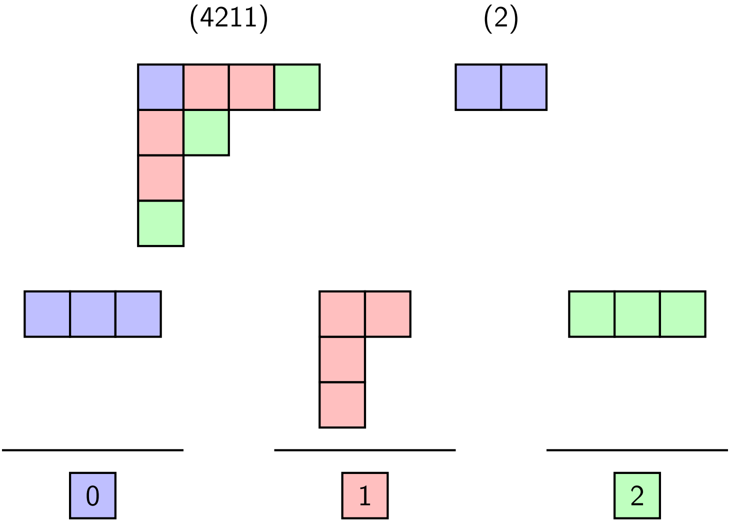

The generalised symmetric group

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$C_r \wr S_n \cong C_r^n \rtimes S_n$

- 'coloured permutations'

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S_4:

1	2	3	4

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- 'coloured permutations'

S_4:	1	2	3	4	$\longrightarrow$	2	1	4	3

The generalised symmetric group

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$C_3 \wr S_4$:

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- 'coloured permutations'

S_4:

1	2	3	4	$\longrightarrow$	2	1	4	3

$C_3 \wr S_4$:

1	2	3	4
1	2	3	4
1	2	3	4

The generalised symmetric group

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- 'coloured permutations'

S_4:

1	2	3	4

$\longrightarrow$

2	1	4	3

$C_3 \wr S_4$:

1	2	3	4
1	2	3	4
1	2	3	4

$\longrightarrow$

2	1	4	3
2	1	4	3
2	1	4	3

The generalised symmetric group

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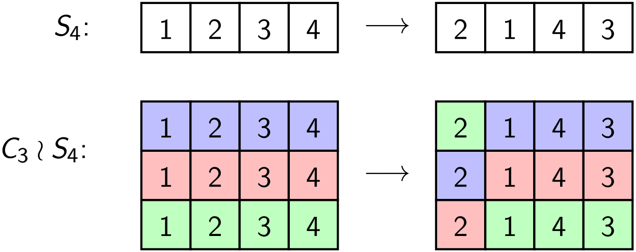

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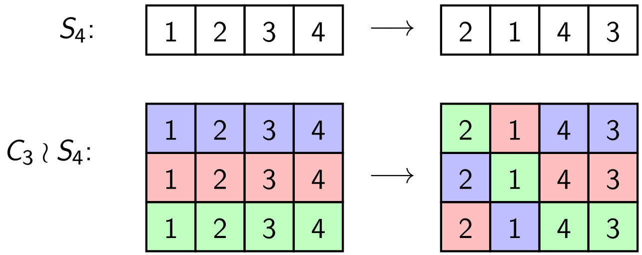

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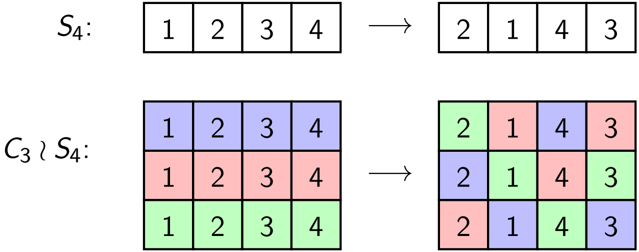

Construction

Construction

λ-transitive set in S_n

Construction

λ-transitive set in S_n + orthogonal array

Construction

λ-transitive set in S_n + orthogonal array = $\underline{\lambda}$-transitive set in $C_r \wr S_n$

Construction

λ-transitive set in S_n + orthogonal array = $\underline{\lambda}$-transitive set in $C_r \wr S_n$

1	2	3

1	2	3
2	3	1
3	1	2
2	1	3
3	2	1
1	3	2

Construction

λ-transitive set in S_n + orthogonal array = $\underline{\lambda}$-transitive set in $C_r \wr S_n$

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(111)

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1	2	3

1	2	3
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3	1	2
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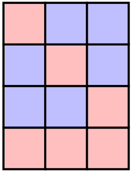
3	2	1
1	3	2

$+$



(111)

Construction

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1	2	3

1	2	3
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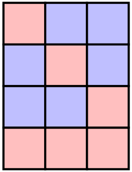
$+$



$=$

1	2	3
1	2	3
1	2	3
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(111)

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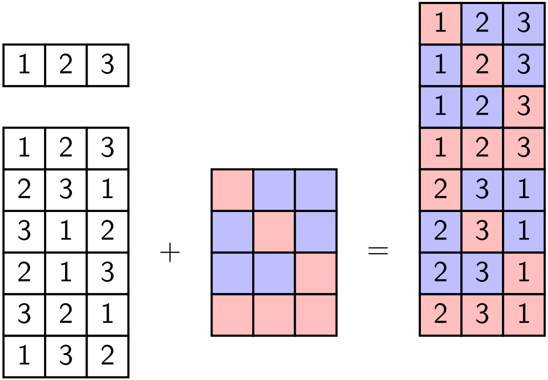

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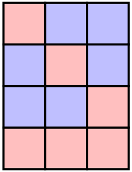

$=$

1	2	3
1	2	3
1	2	3
1	2	3
2	3	1
2	3	1
2	3	1
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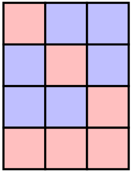

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1	2	3
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1	2	3
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2	3	1
2	3	1
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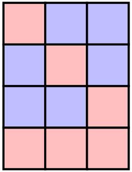

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1	2	3
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1	2	3
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3	1	2
3	1	2
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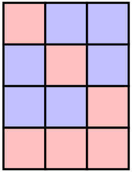

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1	2	3
1	2	3
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3	1	2
3	1	2
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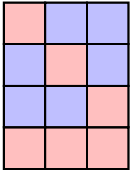

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1	2	3
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3	1	2

2	1	3
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$(11, 1)$