The Saturation Spectrum of Odd Cycles

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Preliminaries

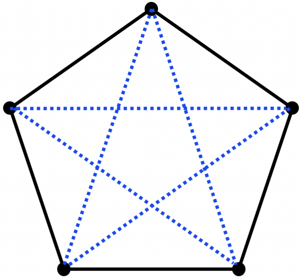
- $G = (V, E)$ has
  - no loops or multiple edges,
  - *order* $n = |V(G)|$ and
  - *size* $m = |E(G)| = e(G)$.
- C_n = cycle on n vertices.
- K_n = complete graphs on n vertices.
- $K_{s,t}$ = complete bipartite graph with parts of size s, t.
- $T_p(n)$ = balanced complete p-partite graph on n vertices.

H-saturated graphs

Definition

Given a graph H, we say that a graph G is *H-saturated* (or *maximal H-free*) if it does not contain an H-subgraph, but the addition of any new edge creates at least one copy of H.

- K_n is the only H-saturated graph for $n < |V(H)|$.
- We use $\mathrm{sat}(n, H)$ for minimum size and $\mathrm{ex}(n, H)$ for maximum size of an H-saturated graph on n vertices.



Maximum m for K_3-saturated graphs

- If G is a K_3 - free graph on n vertices, then $m \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ (Mantel's theorem, 1907)

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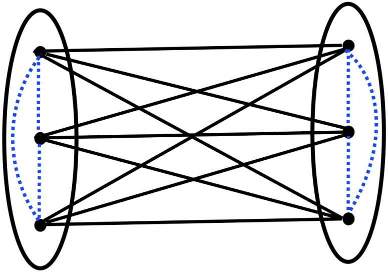
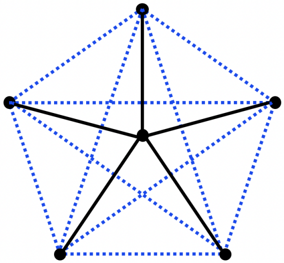
- Turán graph $T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the unique graph that achieves $m = \mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$



Minimum m for K_3-saturated graphs

- $m = \mathrm{sat}(n, K_3) = n - 1$ is achieved only by the star $K_{1,n-1}$.



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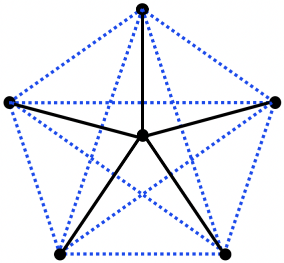

- K_3-saturated graphs are connected.
- K_3-saturated graphs have diameter 2.

All m for K_3-saturated graphs

For which $n-1 < m < \left\lfloor \frac{n^2}{4} \right\rfloor$ are there K_3-saturated graphs?

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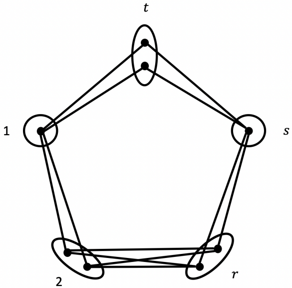
Theorem (Barefoot, Casey, Fisher, Fraughnaugh, Harary, 1994)

- *C_5 - blow up works for $2n-5 \leq m \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$.*
- *No K_3 - saturated graphs for $n-1 < m < 2n-5$ and $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1 < m < \left\lfloor \frac{n^2}{4} \right\rfloor$, except $K_{s,n-s}$.*



All m for K_3 – saturated graphs

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- K_3-saturated graphs with $m > n-1$ are 2-connected.
- 2-connected K_3-saturated graphs must have $m \geq 2(n-4)+3 = 2n-5$ (unbalanced C_5-blowup.)
- Non-bipartite K_3-saturated graphs contain induced odd C_k:

$$m \leq k + \frac{k-1}{2}(n-k) + \left\lfloor \frac{(n-k)^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2-(k-3)^2}{4} \right\rfloor + 2 \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Extremal K_p-saturated graphs

Theorem (Turán, 1941)

$\mathrm{ex}(n, K_p) = e(T_{p-1}(n))$ *achieved only by the* **balanced** *complete* $(p-1)$*-partite graph* $T_{p-1}(n)$.

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Theorem (Erdős, Hajnal, Moon, 1964)

$\mathrm{sat}(n, K_p) = (p-2)(n-p+2) + \binom{p-2}{2}$ *achieved only by the* **unbalanced** *complete* $(p-1)$*-partite graph* $K_{1,1,1,\ldots,1,n-p+2}$.

All m for K_p-saturated graphs

Theorem (Amin, J. Faudree, Gould, 2012)

K_4-saturated graphs of size m exist if and only if
$3n - 8 \leq m \leq \frac{n^2-n+4}{3}$ *or*
$m = e(K_{s,t,n-s-t})$ *for* $s, t \geq 1$.

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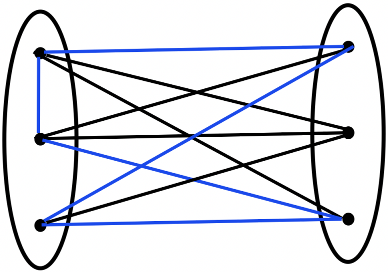
Theorem (Amin, J. Faudree, Gould, Sidorowicz, 2013)

K_p-saturated graphs of size m exist if and only if
$(p-1)(n - \frac{p}{2}) - 2 \leq m \leq \left\lfloor \frac{(p-2)n^2 - 2n + p - 2}{2(p-1)} \right\rfloor + 1$ *or*
$m = e(G)$ *for G complete $(p-1)$-partite.*

Maximum m for C_5 - saturated graphs

$K_{s,n-s}$ is C_5-saturated when $s, n-s \geq 3$.

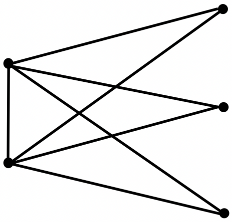


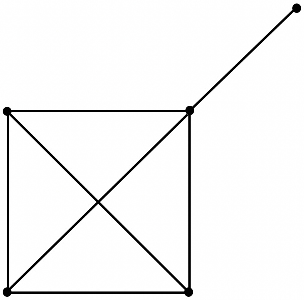
Theorem (Bollabás, 1978)

If $n \geq 6$, *then* $\mathrm{ex}(n, C_5) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Maximum m for C_5 - saturated graphs, $n = 5$

Exactly two graphs achieve $\mathrm{ex}(5, C_5) = 7$:





Minimum m for C_5 - saturated graphs

Theorem (Chen, 2009)

If $n \geq 5$, then $\mathrm{sat}(n, C_5) = \left\lceil \frac{10(n-1)}{7} \right\rceil - \epsilon$, *where*

$$\epsilon = \begin{cases} 1, & \textit{for} \quad n = 11, 12, 13, 14, 16, 18, 20 \\ 0, & \textit{otherwise.} \end{cases}$$

All m for C_5 - saturated graphs

Theorem (Gould, Kang, K, 2023)

For $n \geq 9$, there is a C_5-saturated graph on m edges if and only if

$$sat(n, C_5) \leq m \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 6 \quad or$$

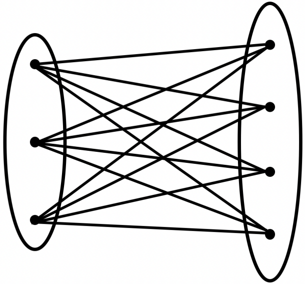
$$m = s \cdot (n - s) \quad or$$

$$m = s \cdot (n - s - 2) + 3.$$

- Non-existence proof for C_5-saturated graphs is more involved than for C_3-saturated graphs (induction)

All m for C_5 - saturated graphs

- $m = s \cdot (n - s)$



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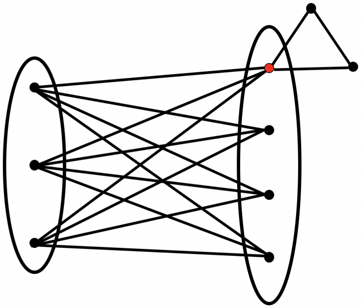
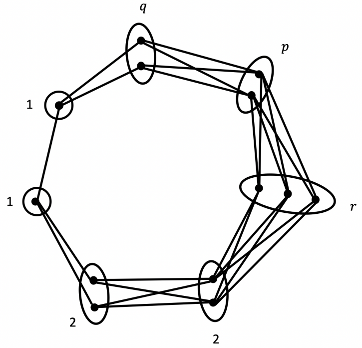
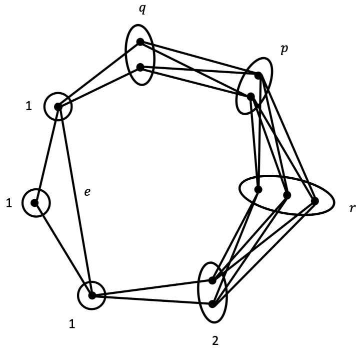


All m for C_5 - saturated graphs

C_7 blow-up provides values for $3n - 15 \leq m \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 3$

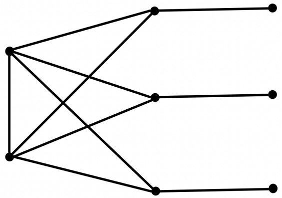
- $C_7(1, 1, 2, 2, r, p, q)$
- $C_7(1, 1, 1, 2, r, p, q) + e$



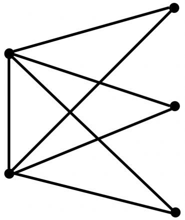


Definition

For $c \geq 1$ and positive integers $n_0, n_1, \ldots, n_c$, let $H(n_0, n_1, \ldots, n_c)$ be the graph obtained from $c+1$ cliques $V_0, V_1, \ldots, V_c$ with $|V_i| = n_i$ by making every vertex in V_0 adjacent to a fixed vertex $v_i \in V_i$ for all $1 \leq i \leq c$.



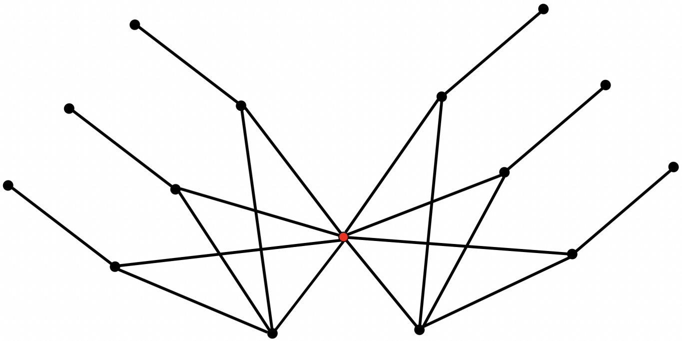
$H(2,2,2,2)$



$H(2,1,1,1) = K_{1,1,3}$

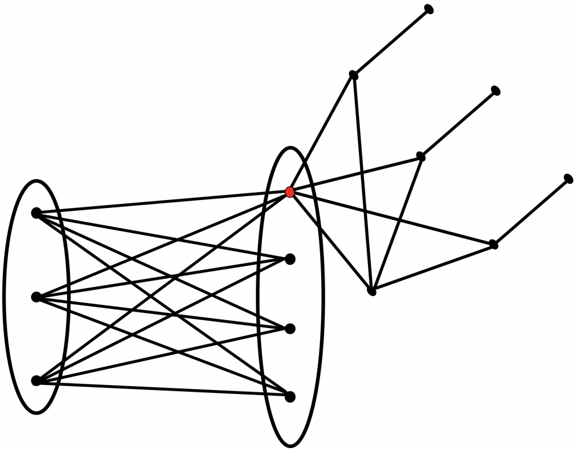
All m for C_5 - saturated graphs

- $\mathrm{sat}(n, C_5) \leq m \leq 2n - 3$



All m for C_5 - saturated graphs

- $2n - 2 \leq m \leq 3n - 16$



C_{2k+1}-saturated graphs

Theorem (Füredi, Gunderson, 2015)

If $n \geq 2k-2$, then $\mathrm{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Also characterized extremal graphs for all n.

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Theorem (Füredi, Kim, 2013)

$\mathrm{sat}(n, C_{2k+1}) \leq n + \frac{n}{2k-3} + O(k^2)$.

The exact value of $\mathrm{sat}(n, C_{2k+1})$ is unknown for $2k+1 \geq 7$, but Füredi, Kim conjecture that their construction is optimal.

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Theorem (Gould, Kang, K, 2023)

If $n \geq 6k-3$, then there is a C_{2k+1} - saturated graph G on n vertices and m edges if

$$\frac{k+1}{2}n - k \leq m \leq \left\lfloor \frac{(n-4k+5)^2}{4} \right\rfloor + \binom{2k+1}{2} - 6$$

Even cycles are much harder!

Theorem (Ollman, 1972)

$\mathrm{sat}(n, C_4) = \left\lfloor \frac{3n-5}{2} \right\rfloor.$

Theorem (Lan, Shi, Wang, Zhang, 2021)

$\frac{4n}{3} - 2 \leq \mathrm{sat}(n, C_6) \leq \frac{4n}{3} + \frac{1}{3}.$

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Theorem (Erdős (1965), Bondy-Simonovits (1974))

$\mathrm{ex}(n, C_{2k}) \le ckn^{1+1/k}.$

Open problems

Theorem (Gould, Kang, K, 2023)

A nonbipartite 2-connected C_5-saturated graph has at most $\left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 6$ edges.

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  $M \geq \frac{(n-4k+5)^2}{4}$
- Which graphs have a gapless saturation spectrum?