Constructing witnesses for non-spreading permutation groups

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Perth

Synchronisation heirarchy

A permutation group satisfies:

$$\begin{array}{c} \text{spreading} \\ \Downarrow \\ \text{separating} \\ \Downarrow \\ \text{synchronising} \\ \Downarrow \\ \text{primitive} \end{array}$$

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For example

- primitive: No G-invariant partition.
- imprimitive: $\exists G$-invariant partiton.

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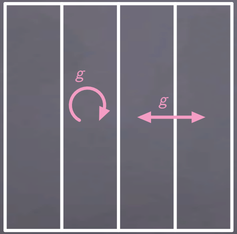
For example

- primitive: No G-invariant partition.
- imprimitive: $\exists G$-invariant partiton.

imprimitive
$\Downarrow$
nonsynchronising
$\Downarrow$
nonseparating
$\Downarrow$
nonspreading

Witnesses

(im)primitive



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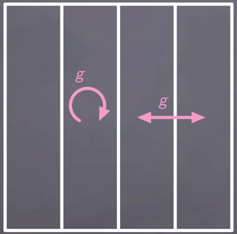
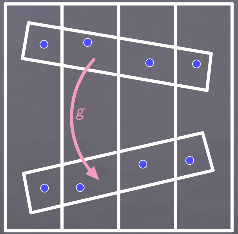


(non)synchronising



Witnesses

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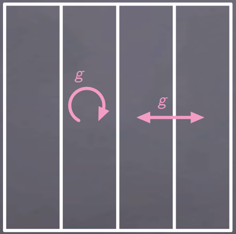
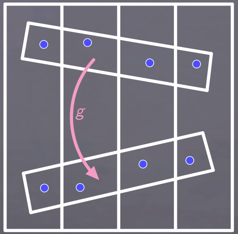
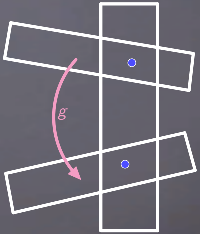


(non)synchronising



(non)separating



Witnesses

Let G be a permutation group acting on the set Ω.

For the following properties, witnesses are given by:

- imprimitive: invariant partition
- non-synchronising: section-regular partition
- non-separating: set A, set B s.t. $|A||B| = |\Omega|$ and $|A \cap B^g| = 1, \forall g \in G$
- non-spreading: multiset A, set B s.t. $|A|$ divides $|\Omega|$ and $|A \star B^g| = \lambda, \forall g \in G$

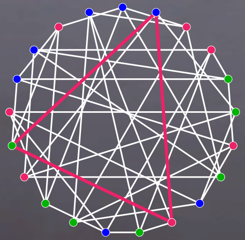
Witnesses using graphs

Theorem

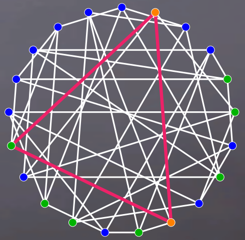
G is non-synchronising if and only if there exists a non-trivial G-invariant graph Γ such that $\omega(\Gamma) = \chi(\Gamma)$.

Theorem

G is non-separating if and only if there exists a non-trivial G-invariant graph Γ such that $\omega(\Gamma)\alpha(\Gamma) = |\Omega|$.



$\chi = \omega = 3$



$\omega \cdot \alpha = 3 \cdot 7 = 21$

Witnesses using graphs

What about spreading? No graph version!

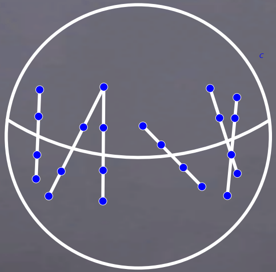
Difficult to find witnesses due to lack of tools.

For example, Aráujo, Cameron, and Steinberg, comment in "Between primitive and 2-transitive: Synchronization and its friends" (2017) that

> *"Pablo Spiga was able to show that* $\mathrm{PSp}(4, \mathrm{p})$ *is non-spreading for* $p = 3, 5, 7$ *by computational methods. The issue is unresolved in general."*

Hemisystems

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure. A hemisystem $H \subset \mathcal{P}$ is such that for every $B \in \mathcal{B}$, precisely half the points of B are in H.



G: $\mathrm{Aut}(\mathcal{S})$ acting on points.
Witness: Hemisystem + point set of a line.

The 'AB-Lemma'

Theorem (Bayens)

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure. Let A and B be two subgroups of $\mathrm{Aut}(\mathcal{S})$ such that

1. *B is a normal subgroup of A,*
2. *A and B have the same orbits on $\mathcal{B}$,*
3. *each A-orbit on $\mathcal{P}$ splits into two B-orbits.*

Then there are 2^n hemisystems admitting B, where n is the number of A-orbits on $\mathcal{P}$.

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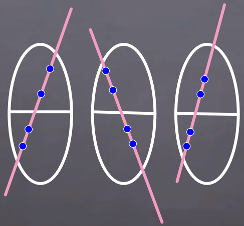

An 'AB-Lemma' type result

Relaxations:

- Sufficient to be locally hemisystem-like
- "exactly half" $\rightarrow$ splits into pieces
- lines point-sets could be anything

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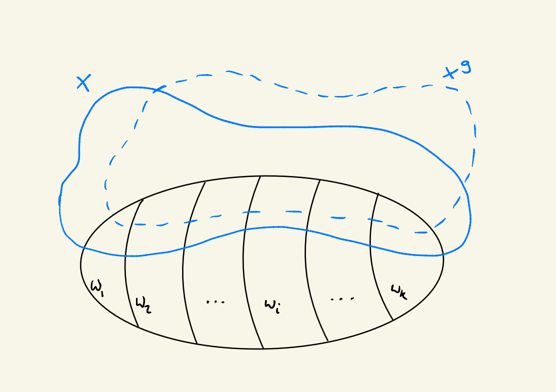
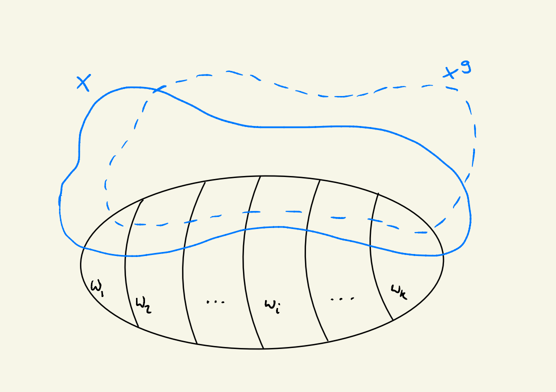
Theorem (Bamberg, Giudici, JL, Royle, 2023+)

Let G be a group acting on Ω, and let A and B be subgroups of G such that $B \triangleleft A$. Let $\omega_1, \ldots, \omega_k \in \Omega$ with $k \geqslant 2$ such that $\omega_i^B \neq \omega_j^B$ for $i \neq j$, and $\omega_1^A = \omega_1^B \cup \omega_2^B \cup \ldots \cup \omega_k^B$. Let $X \subset \Omega$, and $\Delta = \{Y \in X^G \mid Y \cap \omega_1^A \neq \varnothing\}$. If the orbits of A and B on Δ are the same, then $(X, \Omega + k\omega_1^B - \omega_1^A)$ is a witness to G being non-spreading (where $k\omega_1^B$ is the multiset corresponding to ω_1^B with each entry assigned multiplicity k).

An 'AB-Lemma' type result



An 'AB-Lemma' type result



Let $Z \in \Delta$ and $i \in \{1, \ldots, k\}$. Then

$$\begin{aligned}
|Z \cap \omega_1^B| &= |Z \cap (\omega_i^{a_i})^B| && (\omega_i \in \omega_1^A) \\
&= |Z \cap (\omega_i^B)^{a_i}| && (B \lhd A) \\
&= |Z^{a_i^{-1} b_i} \cap \omega_i^B| && (\Delta^A = \Delta^B) \\
&= |Z \cap \omega_i^B|.
\end{aligned}$$

An 'AB-Lemma' type result

If $Z \in X^G \backslash \Delta$ then $|Z \cap \omega_i^B| = 0$ by definition of Δ.
So $|X^g \cap \omega_1^B| = |X^g \cap \omega_i^B|$ for all $g \in G$ and $i \in \{1, \ldots, k\}$.
Hence $k|X^g \cap \omega_1^B| = |X^g \cap \omega_1^A|$.

Let $W = \Omega + k\omega_1^B - \omega_1^A$. Then

- X is a set
- W is a non-trivial multiset (since $k > 1$)
- $|W|$ divides $|\Omega|$
- $|X^g \star W| = |\Omega|$ for all $g \in G$.

So (X, W) is a witness to G being non-spreading.

Applications

With Bamberg, Giudici and Royle, I have applied this technique to find witnesses for many classical group actions. For example

Theorem (Bamberg, JL, Giudici, Royle, 2023+)

For $r > 1$ and even, the permutation groups arising from the action of $\mathrm{PSp}(2r, q)$, $\mathrm{PSU}(2r, q_0)$, and $\mathrm{PO}^+(2r, q)$ on totally isotropic $\frac{r}{2}$-spaces are non-spreading.

It is also a small but important component in the following recent result.

Theorem (Bamberg, Freedman, Giudici, 2023+)

Primitive groups of diagonal type are non-spreading.

Thank you!

Combinatorics in Christchurch
4 - 6 June, 2024

Invited speakers: Bill Martin (keynote), Carmen Amarra, John Bamberg, Gary Greaves, Anita Liebenau, Sho Suda

Organisers: Jesse Lansdown and Geertrui Van de Voorde

Sponsors: The Institute of Combinatorics and its Applications (ICA), NZMS