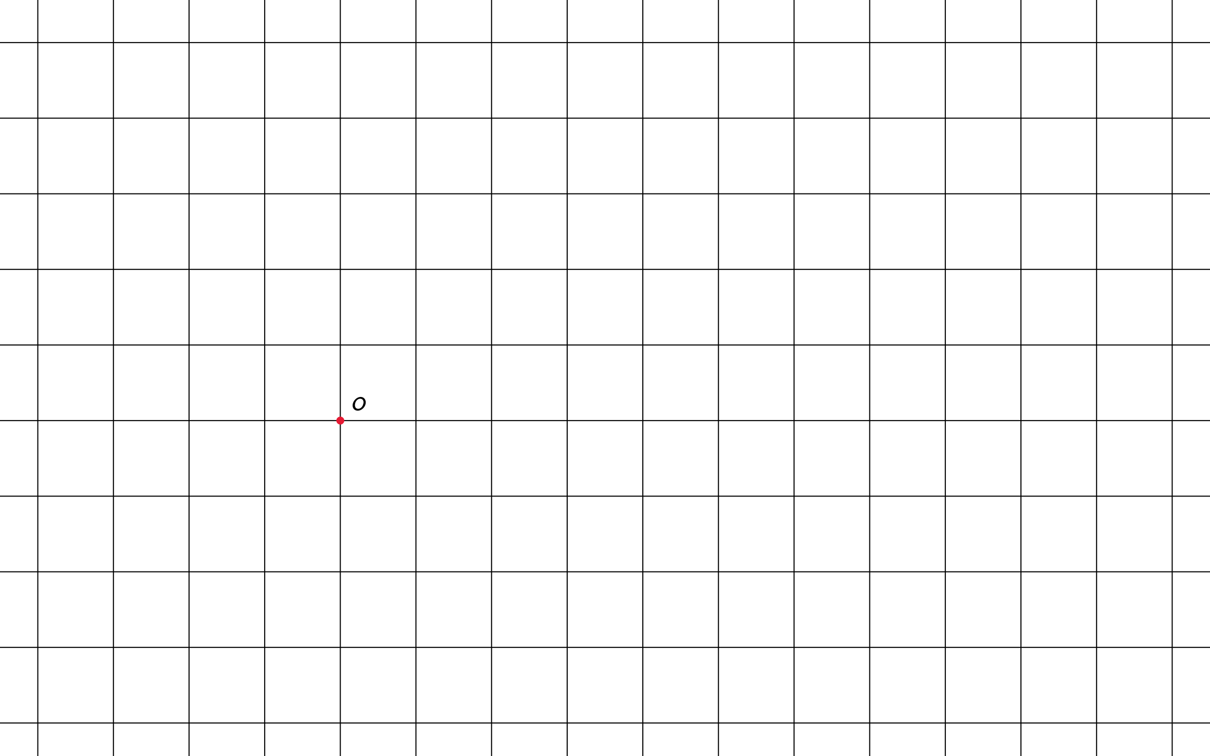
Self avoiding walks on graphs with infinitely many ends

Florian Lehner

joint work with Christian Lindorfer and Christoforos Panagiotis

Mathematicians are like Frenchmen:
whatever you say to them they translate into their own language and forthwith it is something entirely different.

Johann Wolfgang von Goethe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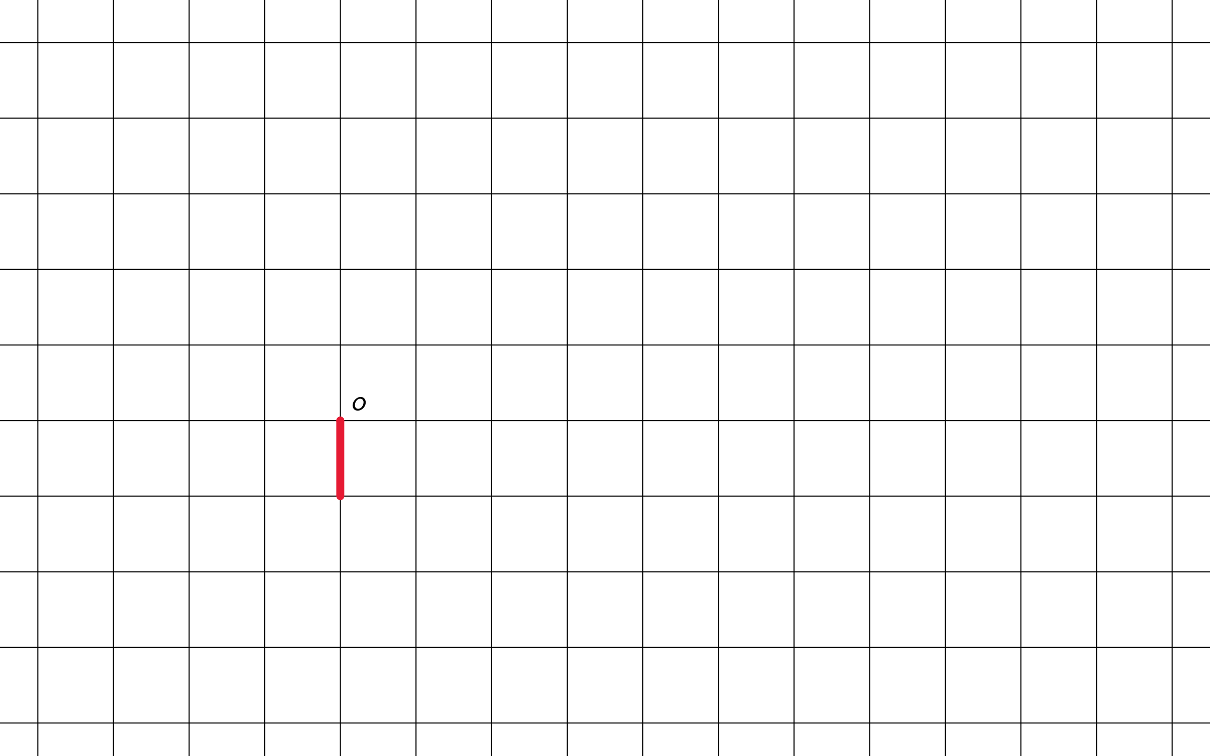
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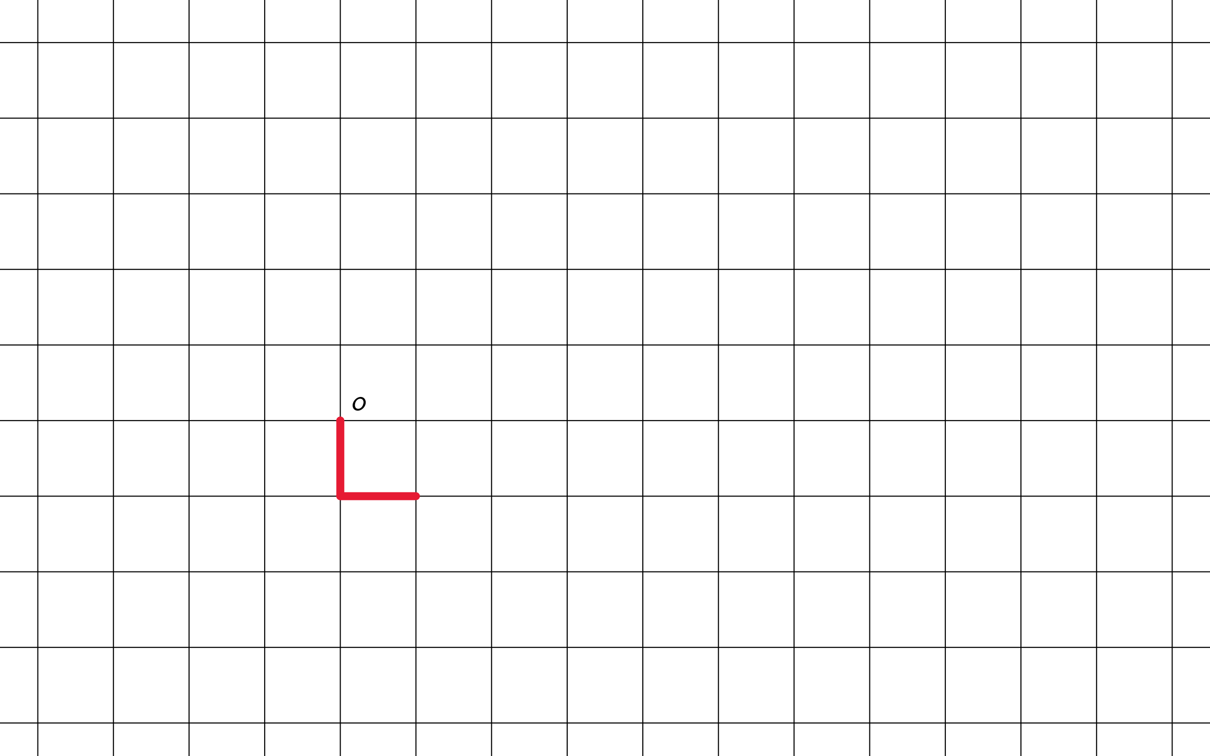

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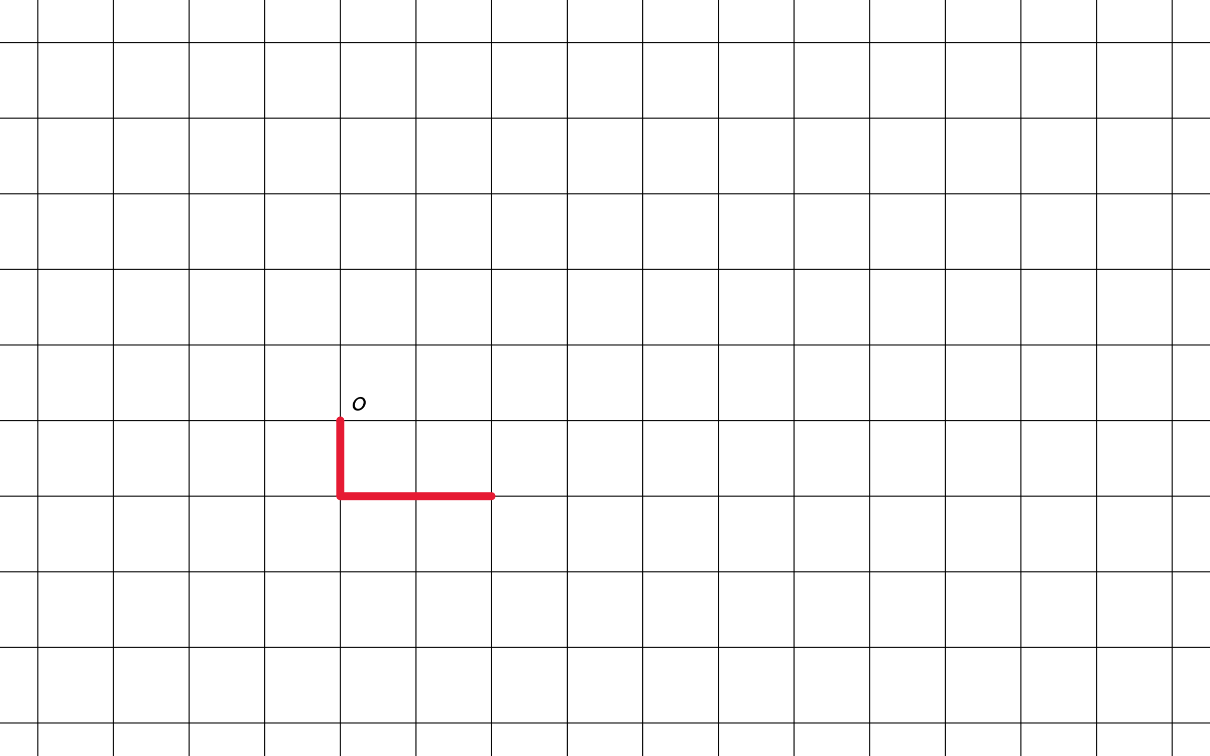
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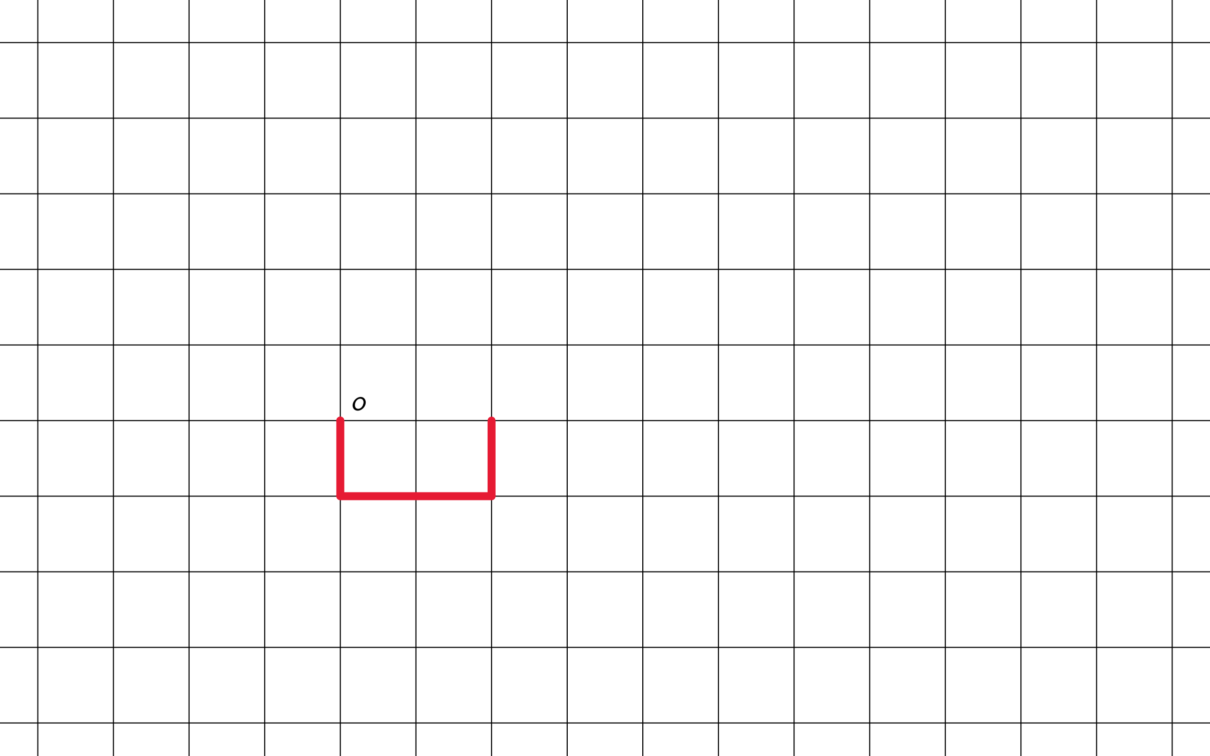
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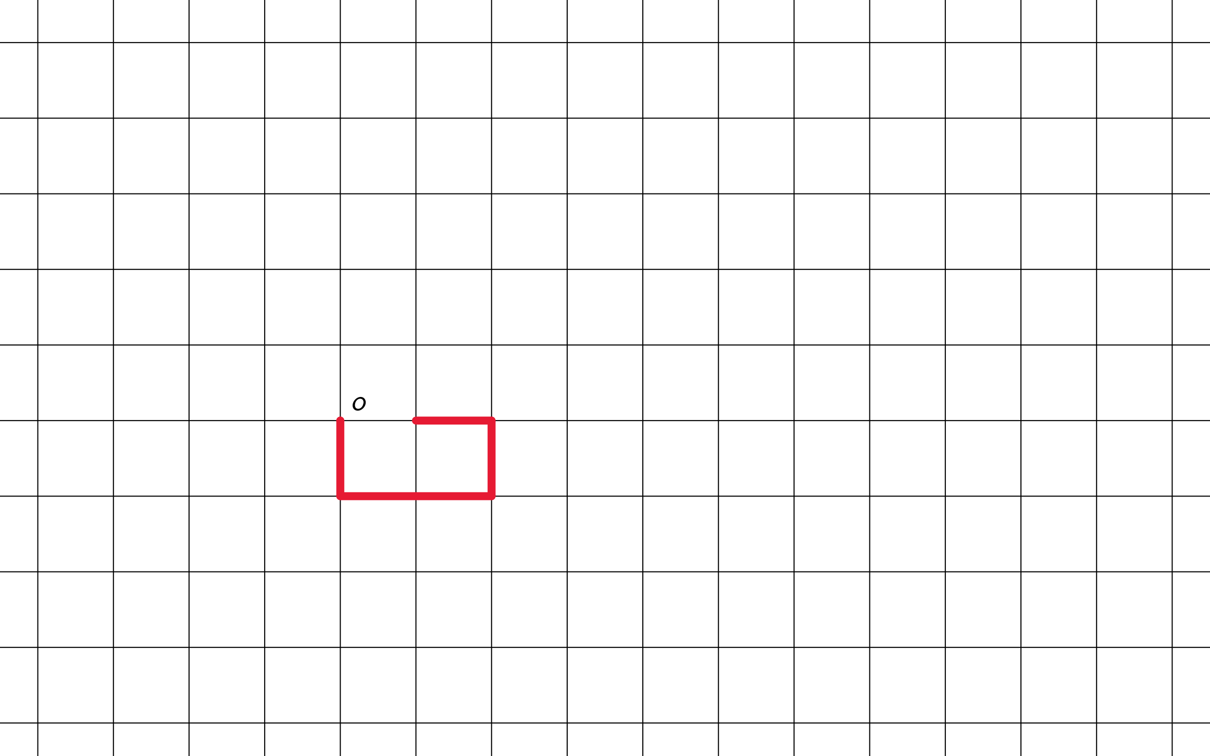
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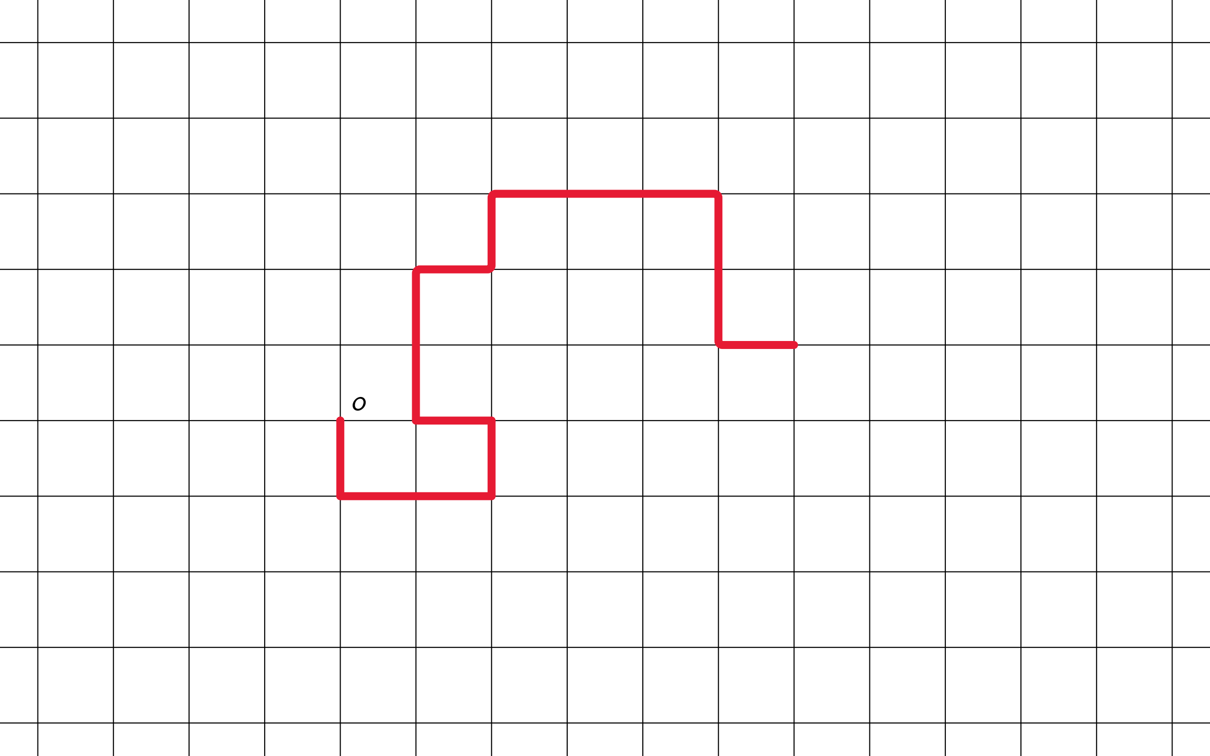
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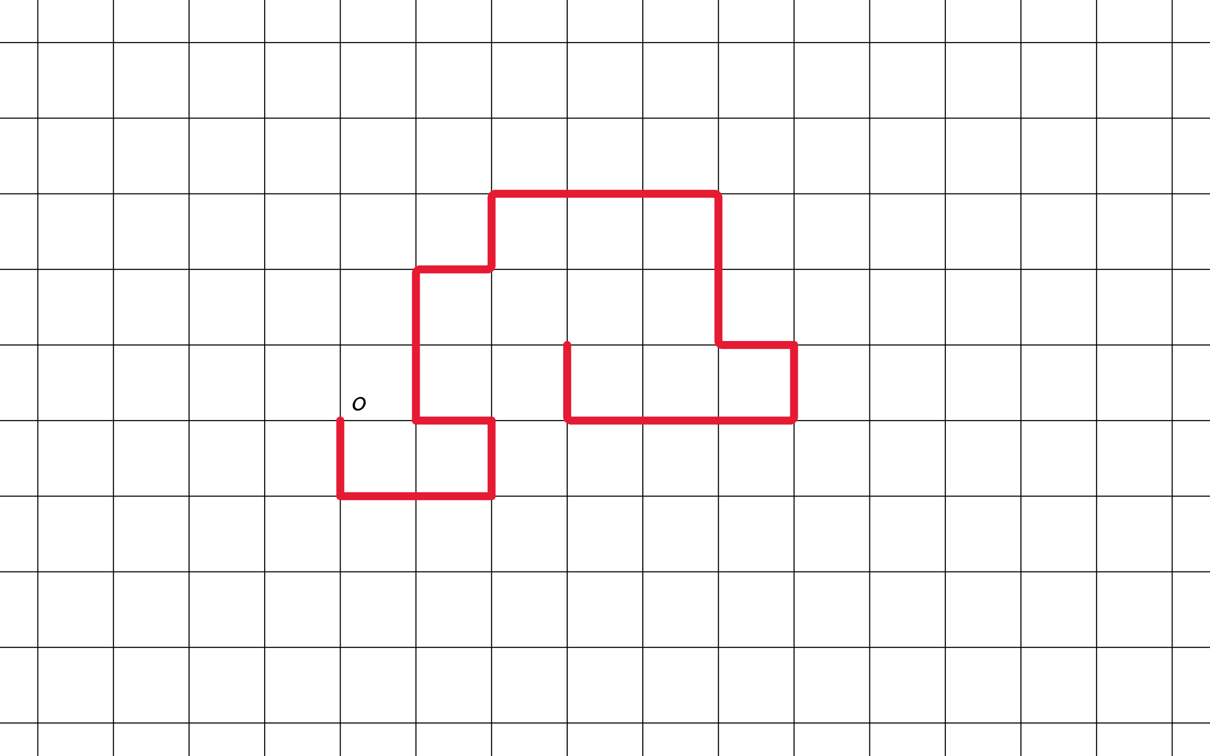
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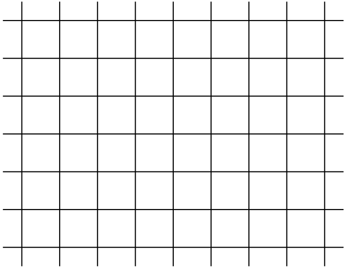
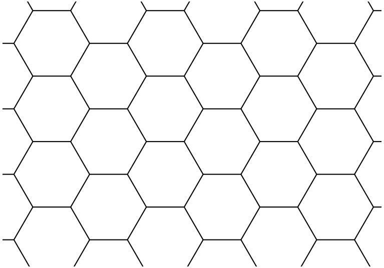
Two basic questions

What is the number c_n of self-avoiding walks of length n?

How far apart are the endpoints of a 'typical' self-avoiding walk?





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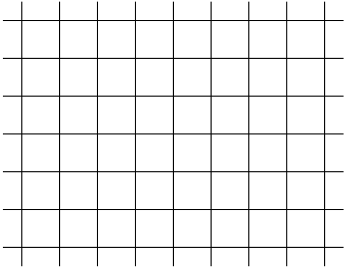
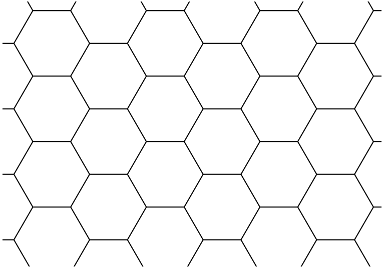


$$2^n \leq c_n \leq 4 \cdot 3^{n-1}$$



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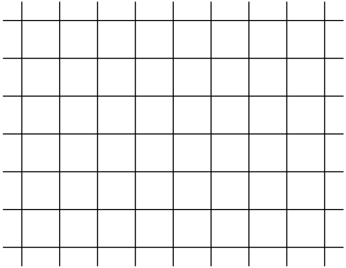

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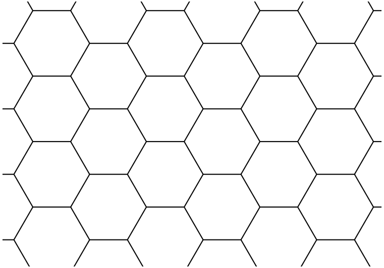

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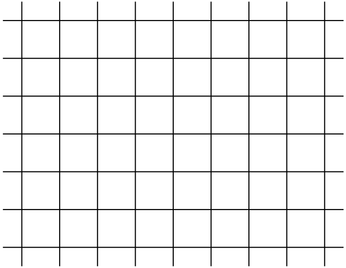
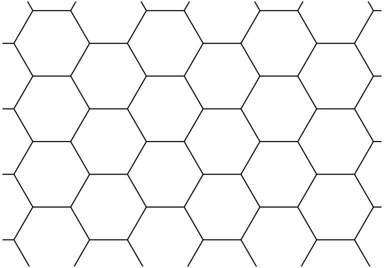


$$2^n \leq c_n \leq 4 \cdot 3^{n-1}$$

$$\mu = \lim_{n\to\infty} \sqrt[n]{c_n}$$



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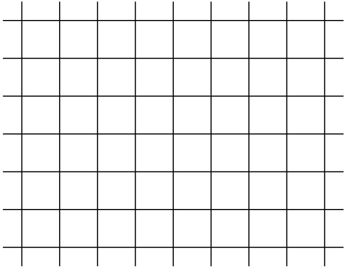

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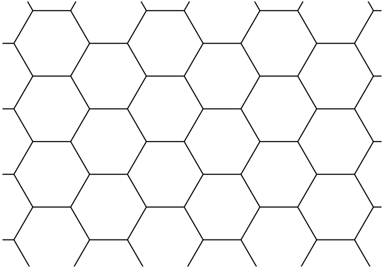
$$\mu = 2.63815853032790\ldots$$



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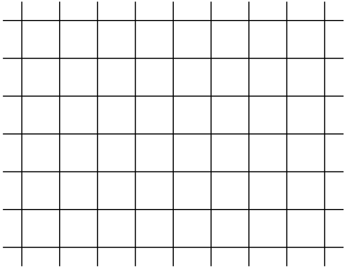

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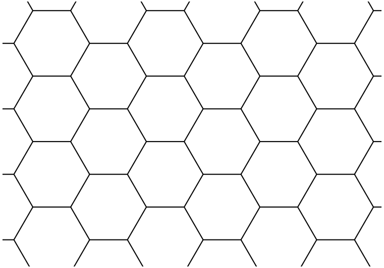

$$\sqrt{2}^n \leq c_n \leq 3 \cdot 2^{n-1}$$

$$\mu = \sqrt{2+\sqrt{2}}$$

Duminil-Copin & Smirnov 2012

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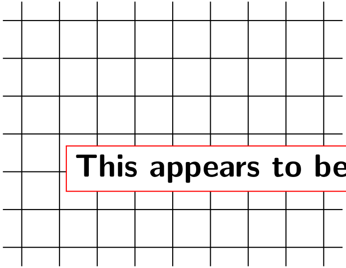

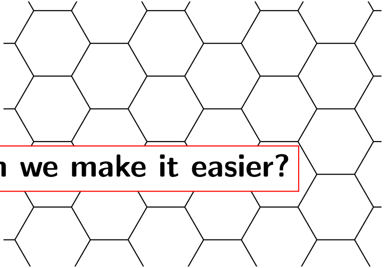

This appears to be very hard, can we make it easier?

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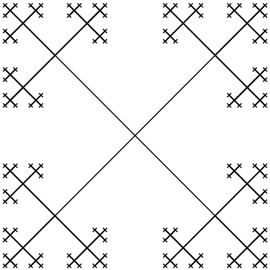

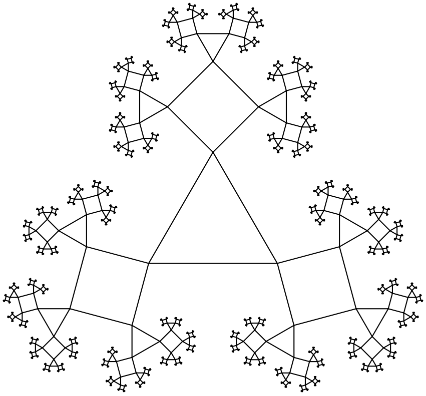

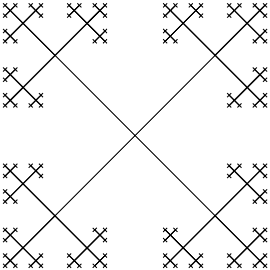

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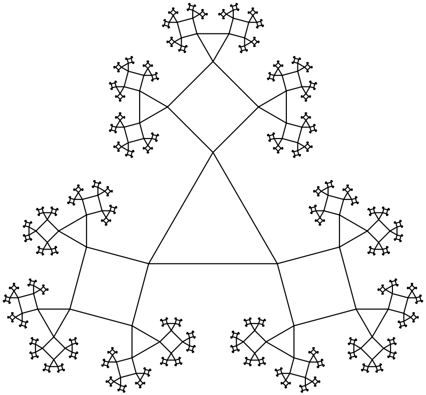
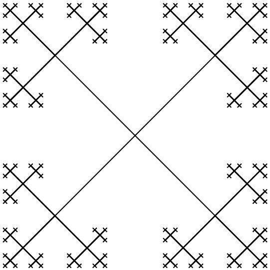
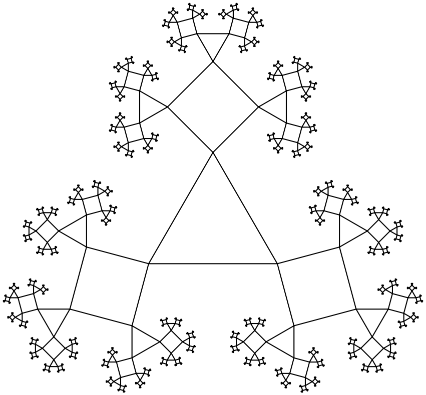
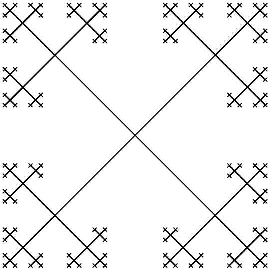




$$c_n = 4 \cdot 3^{n-1}$$
$$\mu = 3$$





$$c_n = 4 \cdot 3^{n-1}$$
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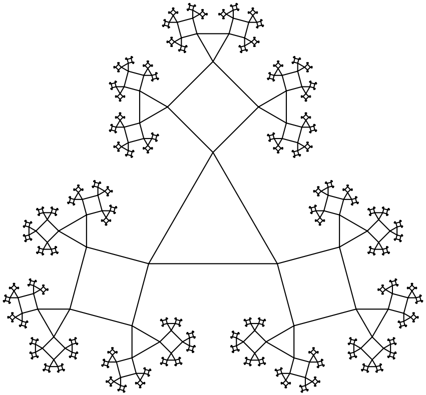
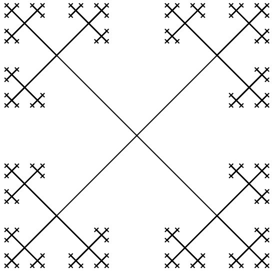
$$\mu = \frac{1+\sqrt{5}}{2}$$

e.g. Alm & Jansen 1990





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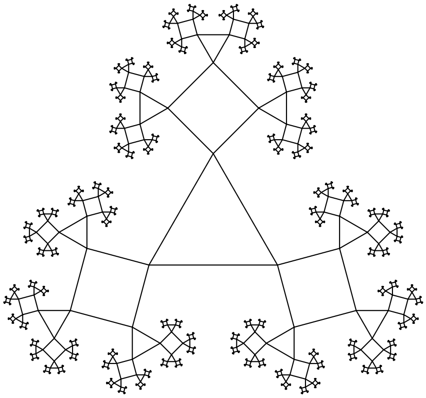


$\mu = \frac{1}{z^*}$, where z^* is the smallest root of
$$4z^5 + 8z^4 + 8z^3 + 4z^2 - 1$$

e.g. Müller & Gilch 2017

Theorem.

Let G be a quasi-transitive, locally finite graph with more than one end.

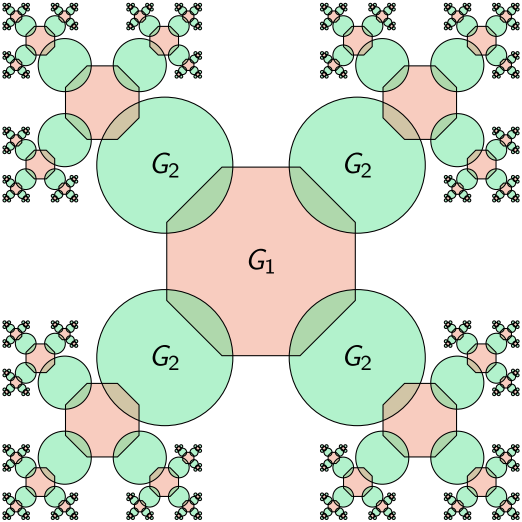
Then G has a 'nice' tree decomposition.

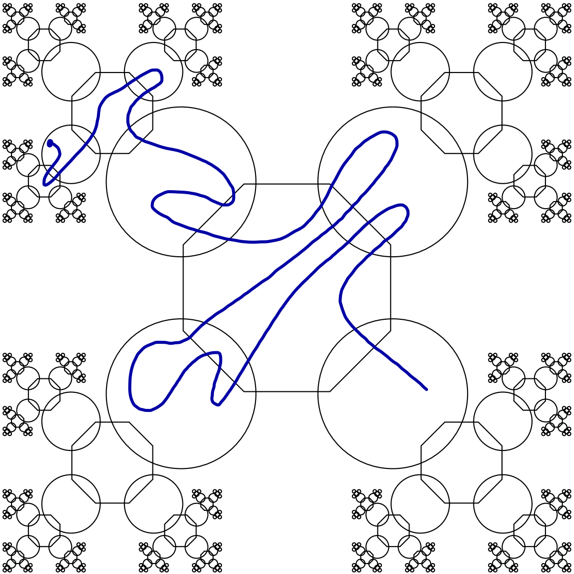
Dunwoody, Krön 2015

Carmesin, Hamann, Miraftab 2021

Hamann, L., Miraftab, Rühmann 2022

L., Lindorfer, Panagiotis 2023+





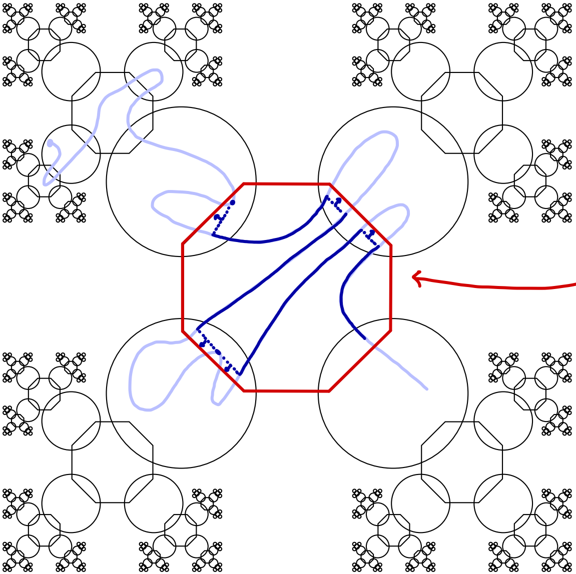
Definition.

The **arrangement** corresponding to a self avoiding walk consists of:

- **shapes** on 'parts'
- **configurations** on 'adhesion sets'

Observation.

There are as many self-avoiding walks as there are **consistent** arrangements.



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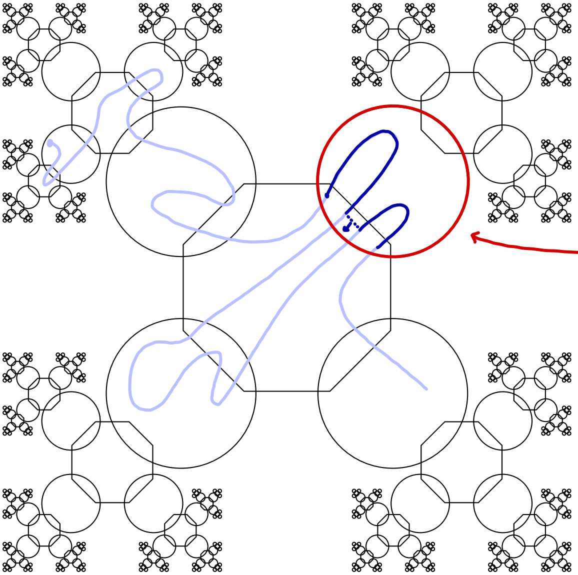

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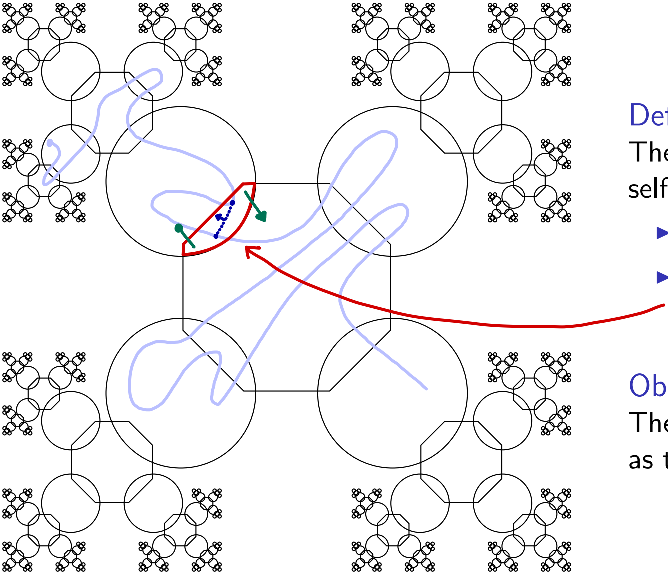

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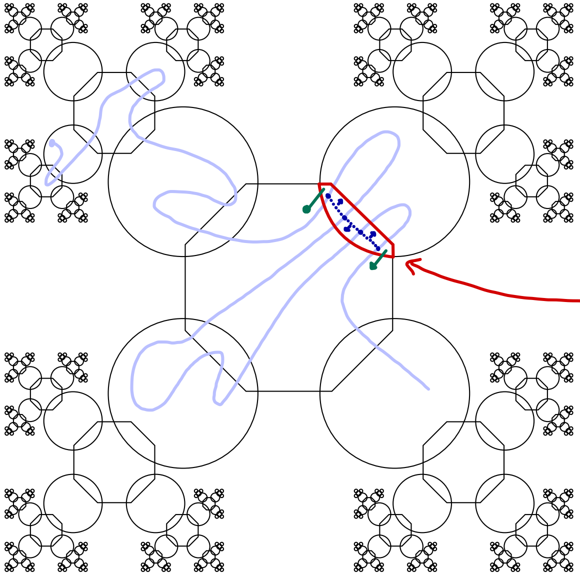

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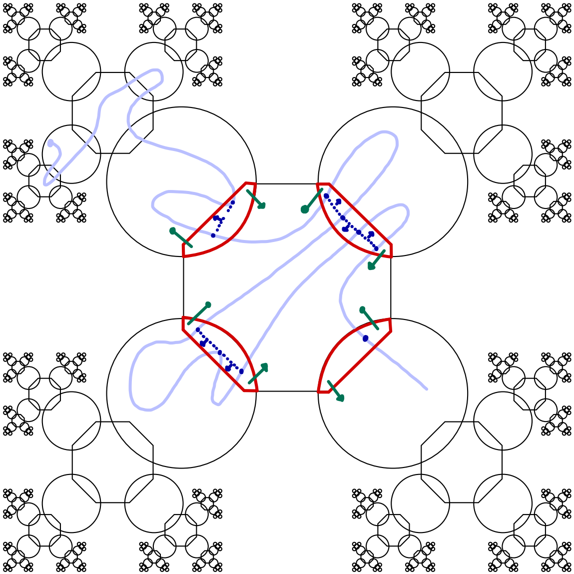

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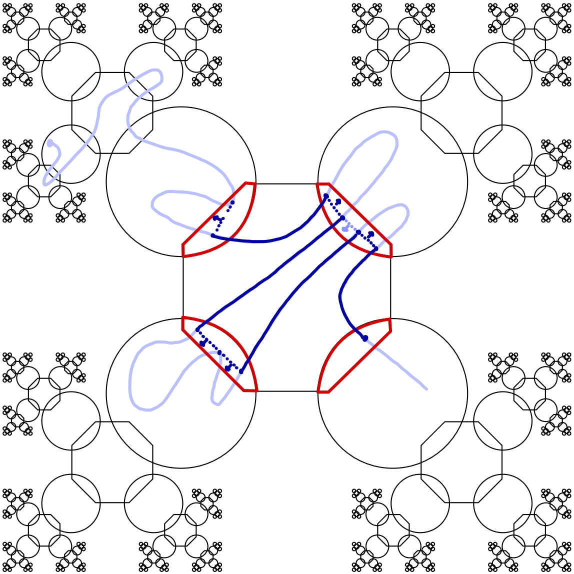

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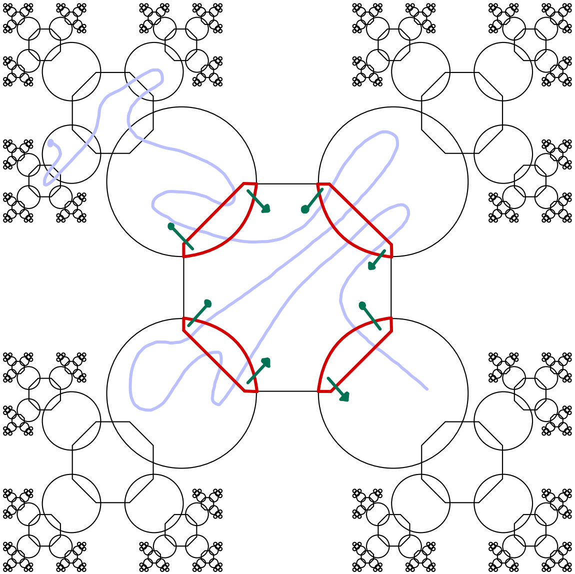

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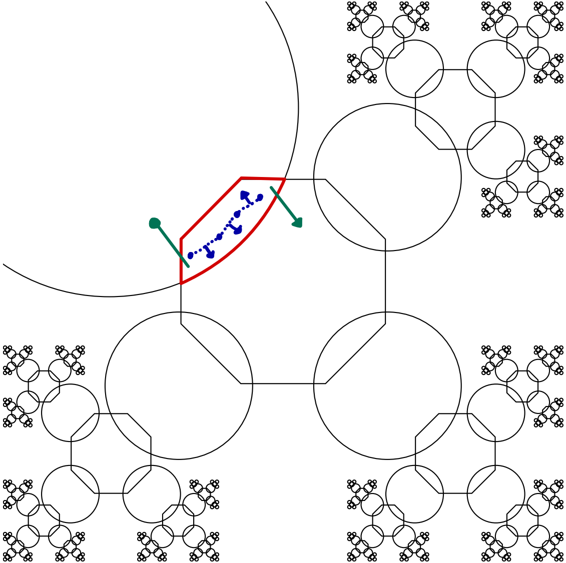
A **completion** of a configuration c is a consistent arrangement on ‘one side’ of the corresponding adhesion set.

Observation.

Let $F_c(z)$ be the generating function counting c-completions of a given length.

Then $\mathbf{F}(z) = (F_c(z))_{c \text{ config.}}$ satisfies a recursion of the form

$$\mathbf{F}(z) = P(z, \mathbf{F}(z))$$



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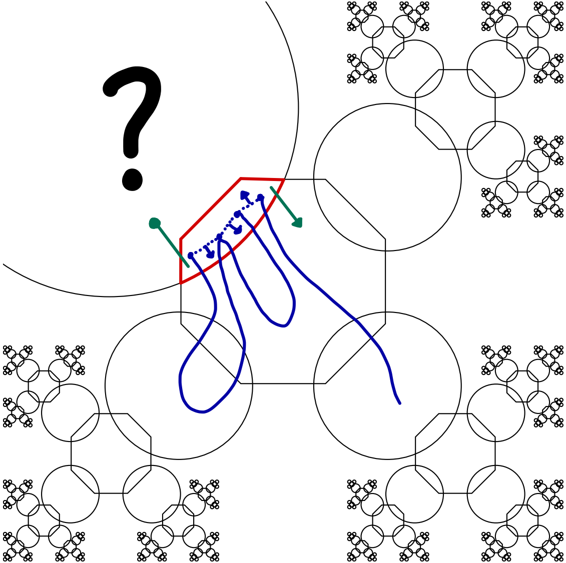

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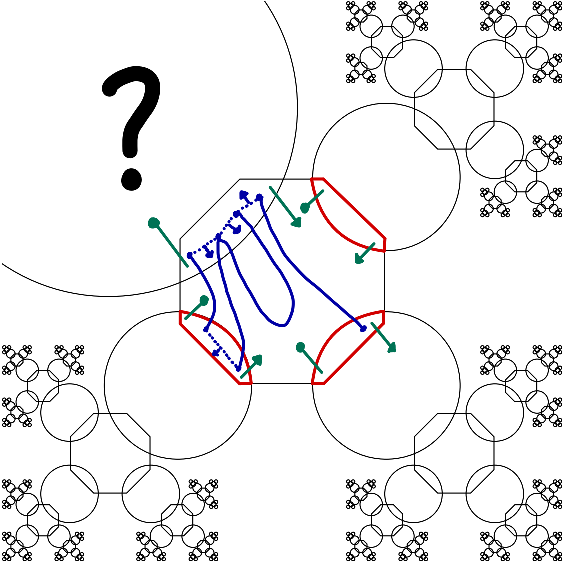

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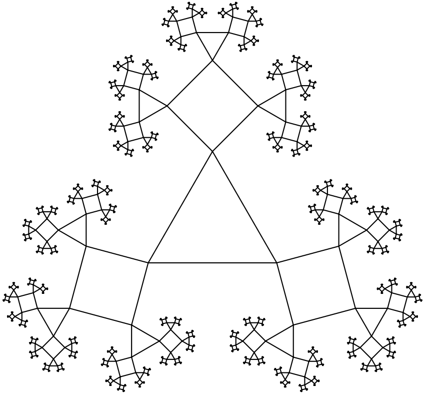
Definition.

The **generating function** of a sequence a_n is the function

$$F(z) = \sum_{i=0}^{\infty} a_n z^n$$

- recursion for a_n gives recursion for $F(z)$
- singularities of $F(z)$ provide information about asymptotics of a_n



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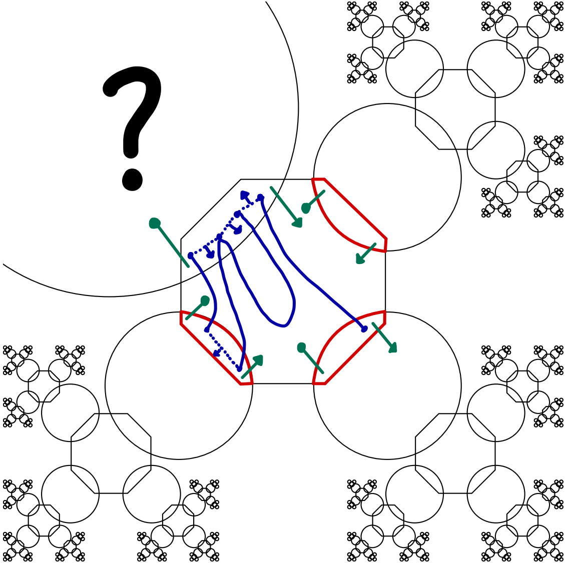

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What is the number c_n of self-avoiding walks of length n?

Theorem.

The generating function $F(z) = \sum_{n \geq 0} c_n z^n$ is algebraic. L. & Lindorfer 2023

Proof.

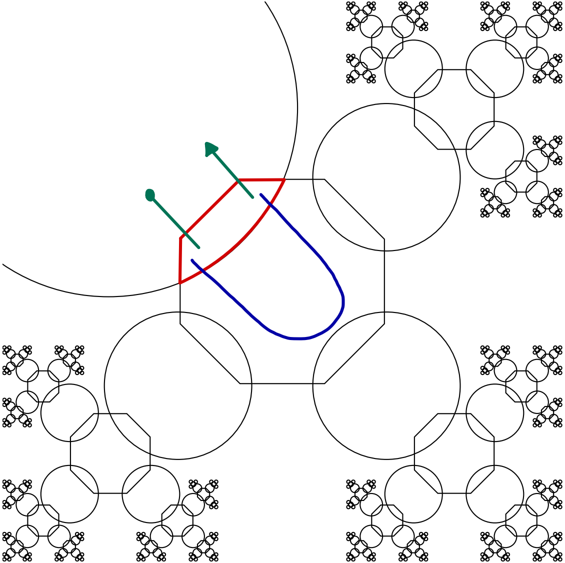
The function P in the recursion $\mathbf{F}(z) = P(z, \mathbf{F}(z))$ is a polynomial.

Moreover, there is some polynomial Q such that $F(z) = Q(z, \mathbf{F}(z))$. $\square$

How far apart are the endpoints of a ‘typical’ self-avoiding walk?

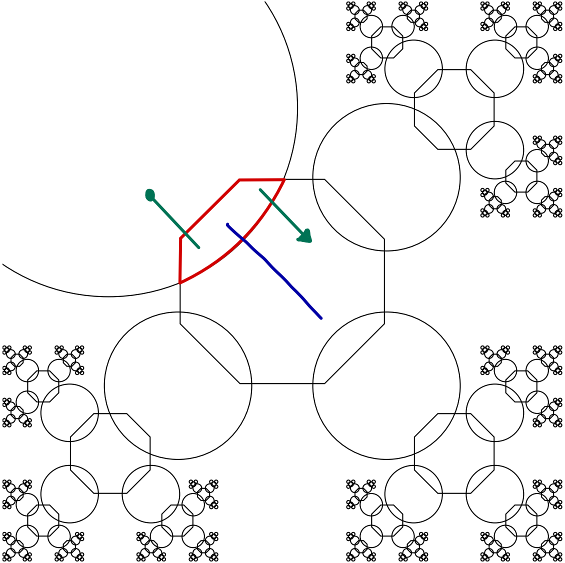
Theorem.
The self avoiding walk on a quasi-transitive graph with more than one end is ballistic, that is, the endpoints are with high probability linearly far apart.

L., Lindorfer, Panagiotis 2023+



Definition.
A configuration is called

- **U-configuration** if entry and exit directions point to the same side
- **I-configuration** if entry and exit directions point to different sides



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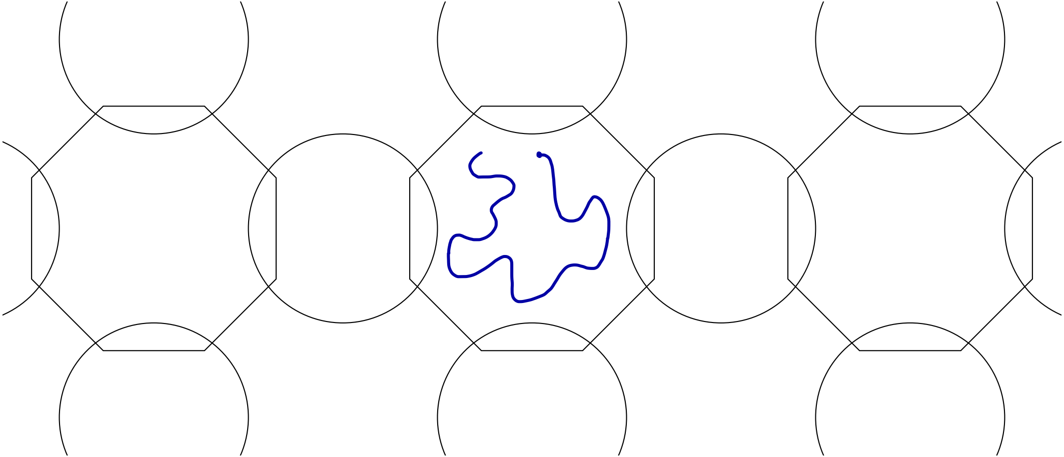
Proof sketch.
Step 1: Show that most self avoiding walks cross at least one adhesion set.

Step 2: Show that I-configurations have more completions than U-configurations. □



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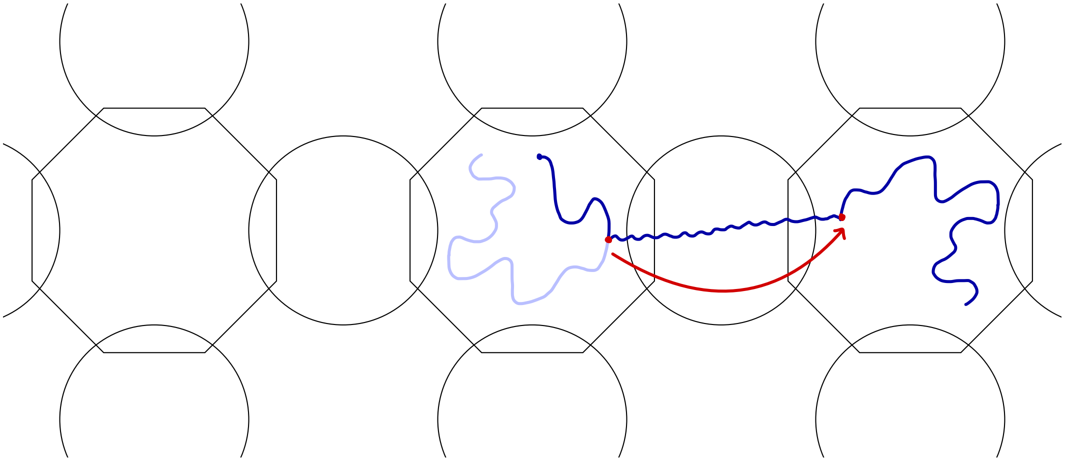

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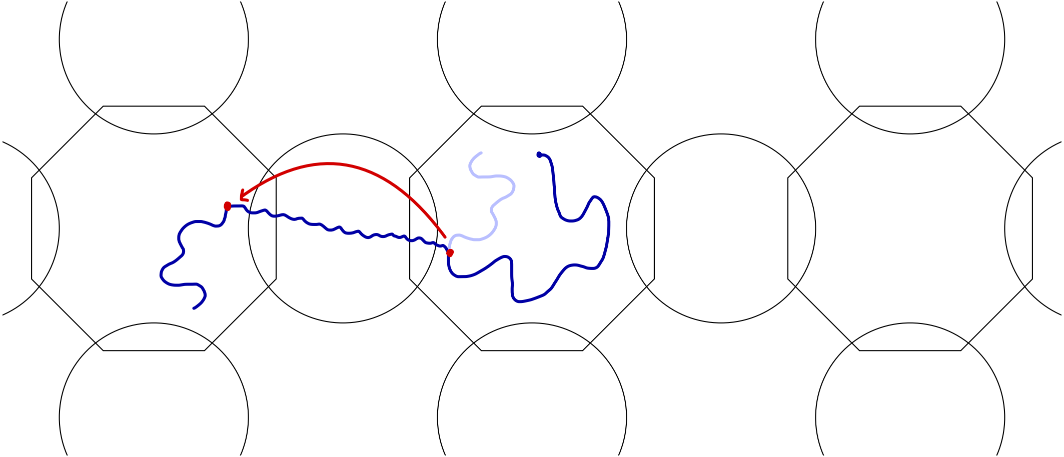

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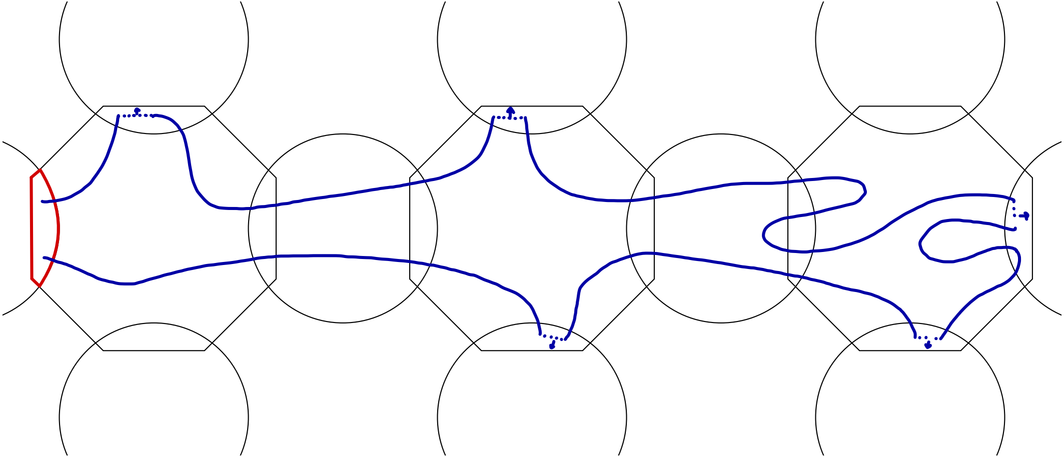
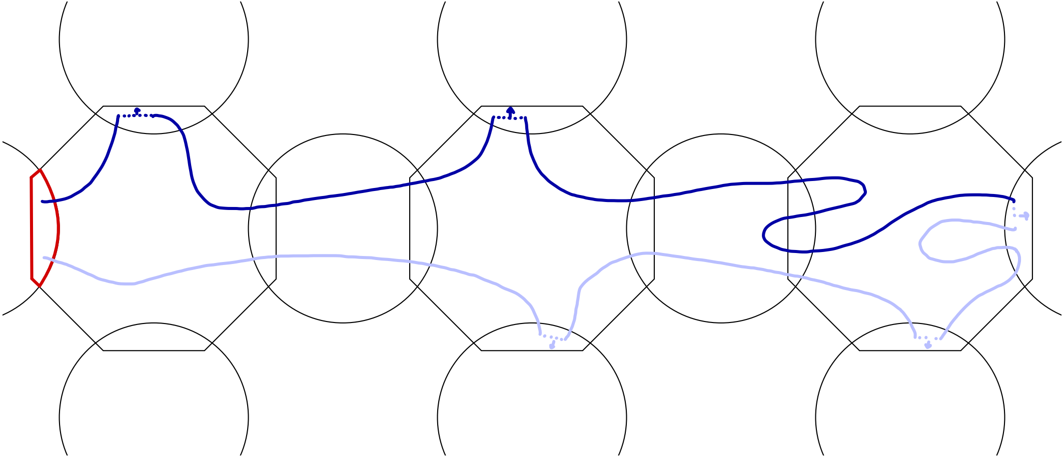
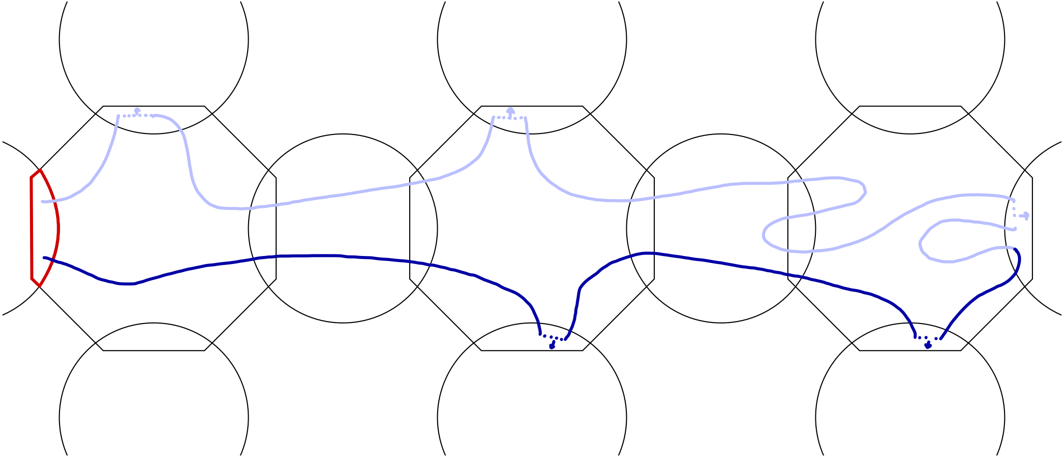


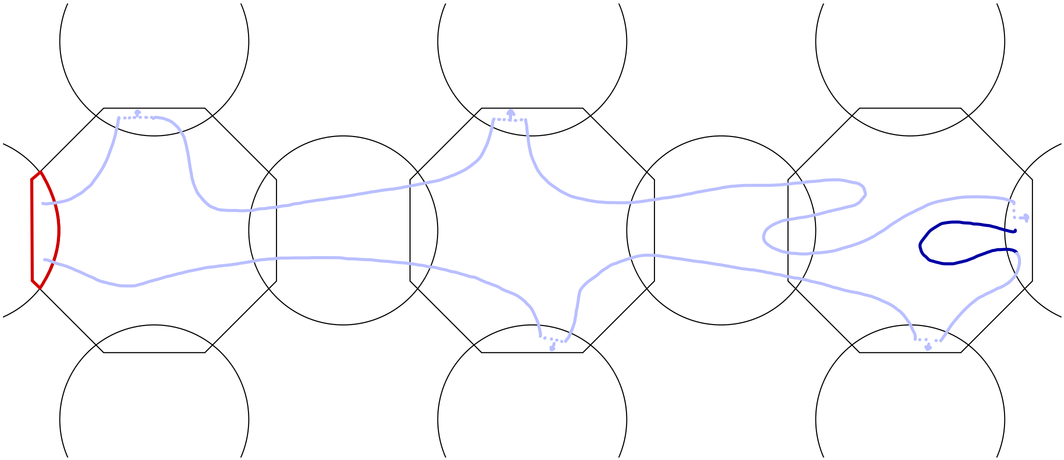
Step 2: Show that I-configurations have more completions than U-configurations.



Step 2: Show that I-configurations have more completions than U-configurations.



Step 2: Show that I-configurations have more completions than U-configurations.



Step 2: Show that I-configurations have more completions than U-configurations.

How far apart are the endpoints of a 'typical' self-avoiding walk?

Theorem.
The self avoiding walk on a quasi-transitive graph with more than one end is ballistic, that is, the endpoints are with high probability linearly far apart.

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Proof sketch.
Step 1: Show that most self avoiding walks cross at least one adhesion set.

Step 2: Show that I-configurations have more completions than U-configurations. □