Tensor representation of semifields and commuting polarities

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General Gist

- Part one:
  - some must-know about finite semifields
  - the Segre and the cyclic models for finite semifields, why Segre has been used more and why the cyclic looks good to us, namely why it deserves more attentions
- Part two:
  - quasi-polar spaces a.k.a. (in this case) **strongly regular graphs** with a nice automorhpism group from finite geometry
  - a nice picture

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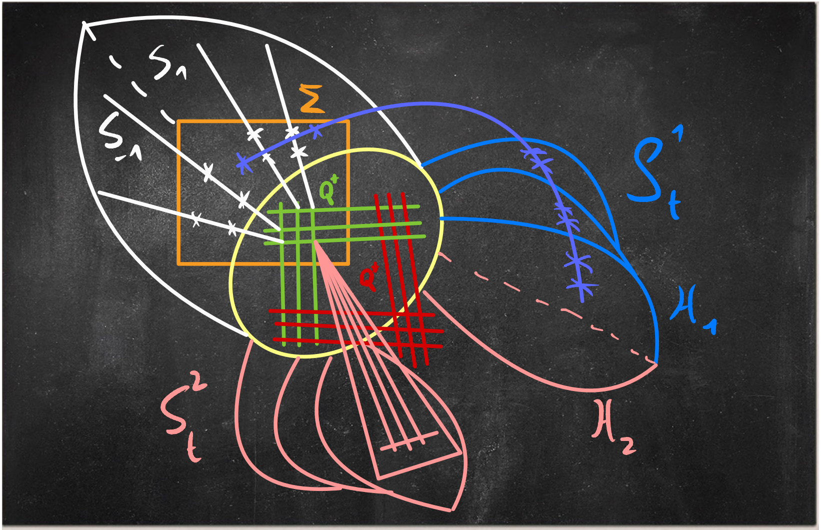
Rorschach test



Finite semi-fields

Definition

A finite semi-field $(\mathbb{S}, +, \circ)$ is a finite not-necessarily commutative, not-necessarily associative division algebra.

Loosely speaking, a finite semifield is a vector space of finite size, in which we can multiply vectors, and we don't allow zero-divisors.
Then we can assume $\mathbb{S} \simeq \mathbb{F}_q^n$, and a common thing is to identify $\mathbb{S}$ with $\mathbb{F}_{q^n}$ and define a new product between elements which coincides with the classical if they are in $\mathbb{F}_q$.

Example

Generalized twisted fields (Albert, 1965): $(\mathbb{F}_{q^n}, +, \circ)$ with

$$x \circ y = xy - cx^{q^i} y^{q^j}$$

$N(c) \neq 1$

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About equivalence

Definition

Two semi-fields $(\mathbb{S}_1, +, \circ)$, $(\mathbb{S}_2, +, \tilde{\circ})$ are equivalent (isotopic) if there are THREE $\mathbb{F}_q$-linear bijections $A, B, C : \mathbb{S}_1 \mapsto \mathbb{S}_2$ such that:

$$A(x \circ y) = B(x) \tilde{\circ} C(y)$$

Why not simply isomorphism?

Semifields correspond to certain translation planes (planes of type V in Lenz classification), and the natural equivalence of these planes correspond precisely to this equivalence.

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Tensor representation of semifields

The set of all bilinear multiplications on $V = \mathbb{F}_q^n$ is $\mathrm{Hom}(V \otimes V, V)$, which is isomorphic to $V^\vee \otimes V^\vee \otimes V$. Namely, **each tensor defines an algebra, and vice-versa.**
The multiplication associated with the tensor $T \in V^\vee \otimes V^\vee \otimes V$ is denoted by $\circ_T$ and defined on the pure tensor $a^\vee \otimes b^\vee \otimes c$ by $x \circ_T y = a^\vee(x) b^\vee(y) c$.

The natural action of Sym(3) on threefold tensors gives rise to the **Knuth orbit**, which permutes the roles of a, b and c, and provides up to six non-isotopic multiplications.

GL(V) × GL(V) × GL(V) acts on $V \otimes V \otimes V$, via

$$(a^\vee \otimes b^\vee \otimes c)^{(f,g,h)} := f(a)^\vee \otimes g(b)^\vee \otimes h(c);$$

namely, on the associated multiplication $(x \circ_T y)^{(f,g,h)} = (x^f \circ_T y^g)^h$.

A more general action is given by the group $\mathrm{GL(V)} \wr \mathrm{Sym}(3)$.

Nota bene

Finite semifields are those algebras without zero-divisors, and correspond to those tensors which are **non-singular**.

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Contractions and non-singularity.

Tensor Contraction

The contraction of $T \in V_1 \otimes \ldots \otimes V_m$ with $u_i \in V_i^*$ is a tensor in $V_1 \otimes \ldots \otimes \hat{V}_i \otimes \ldots \otimes V_m$, defined by

$$u_i(v_1 \otimes \ldots \otimes v_m) = u_i(v_i)v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_m.$$

Contraction Space

The i-th contraction space is the subspace $C_i(T) := \{u_i(T) : u_i \in V_i^*\}$.

Theorem

Two tensors $T_1, T_2 \in V_1 \otimes \ldots \otimes V_m$ are equivalent if and only if their i-th contraction spaces $C_i(T_1)$ and $C_i(T_2)$ are equivalent.

Definition

A vector (1-tensor) is non-singular if it is non-zero. Recursively, a tensor is non-singular if every contraction of it is non-singular.

Contractions and non-singularity in layman terms

Contracting a matrix (2-tensor) means taking a linear combination of its rows/columns (1-tensors). So you get a vector and you have 2 ways to do it. Contracting a t-fold tensor means taking linear combinations of its slices, and you can do it in $t!$ ways. **Instead of studying the tensor you can study the subspace of its contractions "along one direction".**

Is this good news?

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The first Segre of the day

The *(Corrado) Segre embedding* embeds in the natural way $\mathbb{F}_q^n \times \mathbb{F}_q^n \times \mathbb{F}_q^n$ into $\mathbb{F}_q^n \otimes \mathbb{F}_q^n \otimes \mathbb{F}_q^n \simeq \mathbb{F}_q^{n^3}$. Going projective, we see tensors as points of $\mathrm{PG}(n^3-1,q)$. The Segre variety is the set of pure tensors.

Nota bene

The action of GL(V) × GL(V) × GL(V) on tensors induces an action of points of $\mathrm{PG}(n^3-1,q)$ given by **collineations**.

Intermezzo

Corrado Segre is **Beniamino Segre**'s uncle and PhD advisor. The Segre-type theorems Geertrui will talk about after lunch are Beniamino's.

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The cyclic model for threefold tensors

- represent $V \simeq \mathbb{F}_q^n$ as $\mathbb{F}_{q^n}$;
- then the elements $a^\vee, b^\vee \in \mathbb{F}_{q^n}^\vee$, duals of $a, b \in \mathbb{F}_{q^n}$, acts as the trace, $a^\vee(x) = Tr(ax)$
- the multiplication $\circ_T$ defined by $T = a^\vee \otimes b^\vee \otimes c$ becomes:

$$x \circ_T y = Tr(ax)Tr(by)c.$$

- if you write $\hat{v} = (v, v^q, \ldots, v^{q^{n-1}})$ and $M_T = \hat{a}^t \hat{b} c$, the multiplication can be written as

$$x \circ_T y = \hat{x}^{tr} M_T \hat{y}$$

Cyclic model

We can represent *tensors in* $(\mathbb{F}_q^n)^{\otimes 3}$ as *matrices* $n \times n$ over $\mathbb{F}_{q^n}$ and as points of $\mathrm{PG}(n^2-1, q^n)$.

In the cyclic model, the group $\mathrm{GL(V)} \times \mathrm{GL(V)} \times \mathrm{GL(V)}$ does not induce collineations of $\mathrm{PG}(n^2-1, q^n)$. The third $\mathrm{GL}(V)$ is problematic.

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Nota bene

Non-singularity of the matrix of the cyclic model does not mean non-singularity of the tensor.

The trace provides the functional $T^\vee \in \mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}^\vee$, associated with T:

$$\mathrm{T}(x, y, z) = \mathrm{Tr}(x \circ yz) = \mathrm{T}r(\sum_{i,j} \alpha_{i,j} x^{q^i} y^{q^j} z)$$

where $(\alpha_{i,j}) = M_T$.
T is non-singular if and only if this map is non zero for any non zero x, y, z.
Contracting on the third component, we obtain a multiple of a Dickson matrix, namely a matrix of the type

$$\begin{pmatrix} a \\ a^q \\ \dots \\ a^{q^{n-1}} \end{pmatrix} \begin{pmatrix} b & b^q & \dots & b^{q^{n-1}} \end{pmatrix}$$

The set of all contractions forms a subgeometry $\Sigma \simeq \mathrm{PG}(n^2 - 1, q)$.

Theorem (LS-Sheekey)

The contraction space $C_3(T)$ of a tensor T is the unique subspace of Σ of minimal dimension whose extension to Σ^* contains the point $P = \langle T \rangle$.

Takeaways:

- Semifields correspond to **non-singular** tensors of format $n \times n \times n$;
- you can embed these tensors in a projective space in several ways;
- the "natural" embedding is the Corrado Segre (the uncle) embedding
  - good thing: here all the symmetries of the tensors are collineation of the space
  - bad thing: studying non-singularity is still difficult, the contraction spaces do not naturally live in the same space
- the cyclic model has been used by people studying semifields, but not in the context of associated tensors.
  - bad thing: doesn't work for general format of tensors
  - bad thing: **not** all the symmetries of the tensors are collineation of the space
  - very good thing: the contraction spaces live in the same space and have a nice geometric interpretation
  - positive turn of the bad thing: we now understand how the third $\mathrm{GL}(V)$ behaves (fixes the contraction space, and permutes the points in the extension of this space)

Part two: quasi-polar spaces and a nice picture

Definition

Quasi-quadrics and quasi-hermitian varieties (quasi-polar spaces) are sets of points of a projective space with same size and same intersection numbers as the corresponding (non-degenerate) polar space. In particular they are two-character set (w.r.t. hyperplanes).

We like it because

- *few characters is an extremal behaviour.*
- *quadrics and hermitian varieties are probably the nicest objects*, so *if you look like one of them, you look cool.*
- a good deal of interest is due to their link with **strongly regular graphs**.

Linear representation graph

Linear representation graph

Take your two character set $S \subseteq \Sigma = \mathrm{PG}(n-1,q)$ and embed Σ as hyperplane at infinity in $\mathrm{PG}(n,q)$. The graph $\Gamma(S)$ with vertices the affine points of $\mathrm{PG}(n,q)$ and an arc if the line through them has a point of S at infinity is strongly regular with parameters (q^n, k, λ, μ) with $k = |S|(q-1)$, $\lambda = k-1+(k-qw_1+1)(k-qw_2+1)$ and $\mu = k+(k-qw_1)(k-qw_2)$, where w_1, w_2 are the two characters.

Theorem (Cara-Rottey-Van de Voorde 2014)

Under reasonable and mild conditions all the isomorphisms between two linear representation graphs $\Gamma(S_1), \Gamma(S_2)$ *are those coming from collineations.*

As a consequence: new quasi-polar spaces $\Longrightarrow$ new non-isomorphic SRG with the same parameters.
If they also have some automorphisms $\Longrightarrow$ new non-isomorphic SRG with the same parameters and same automorphisms.

Non-singular $2 \times 2 \times 2$ tensors

In the first part we said that a tensor T, represented in the cyclic model by the matrix $(\alpha_{i,j})$, is non-singular if and only if

$$\mathrm{T}(x,y,z) = \mathrm{Tr}(x \circ yz) = \mathrm{Tr}(\sum_{i,j} \alpha_{i,j} x^{q^i} y^{q^j} z) \neq 0$$

for any non zero x, y, z.

For $n = 2$ and $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, this is equivalent to ask that:

$$\begin{cases} Q + Hz + Q^q z^2 \neq 0 \\ z^{q+1} = 1; \end{cases}$$

where $Q = \alpha\delta - \beta\gamma$ and $H = \alpha^{q+1} - \beta^{q+1} - \gamma^{q+1} + \delta^{q+1}$ and $z \in \mathbb{F}_{q^n}$.

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Commuting polarities

Q and H from the system define a quadric $\mathcal{Q}^+$ and an Hermitian surface $\mathcal{H}$.

Why commuting polarities in the title?

$\mathcal{Q}^+$ and $\mathcal{H}$ are in permutable position, namely $\perp_Q$ and $\perp_H$ are commuting polarities.

Two commuting polarities $\perp_1$ and $\perp_2$ determine an involution $\rho = \perp_1 \perp_2$ which in turn fix **a subgeometry** Σ, which coincides with the one given by Dickson matrices of the cyclic model.

$$\Sigma \cap \mathcal{H} = \Sigma \cap \mathcal{Q}^+ = \mathcal{Q}_0^+$$

$\mathcal{Q}_0^+$ is an hyperbolic quadric of the subgeometry Σ. If we look at the orbits of (a subgroup of) the stabiliser of $\mathcal{Q}_0^+$, a lot of interesting geometry arises.
The group G we want is isomorphic to $\mathrm{PCSO}^+(4, q)$ and consists of collineations of the form

$$v \mapsto AvB \quad \text{or} \quad X \mapsto Av^T B,$$

where A and B are invertible 2×2 Dickson matrices with coefficients in $\mathbb{F}_{q^2}$ whose determinant is a square of $\mathbb{F}_q$.

Study of the system

Lemma (SL, John Sheekey)

$P = (\alpha, \beta, \gamma, \delta)$ is non-singular if and only if $\Delta(P) \in \Box_q^\times$, where $\Delta = H^2 - 4Q^{q+1}$.

Nota bene:

P non-singular if and only if each of it's contractions non-singular. In $\mathrm{PG}(3, q^2)$ each point lies on precisely one extended Σ-line.
This implies that the contraction $C(P)$ is that unique line.

Corollary-SL, Sheekey

Non-singular points P are points not in Σ, lying on extended sublines external to $\mathcal{Q}_0^+$.
Equivalently, non-singular points are points not in Σ and not on extended $\mathcal{Q}_0^+$-tangent planes.

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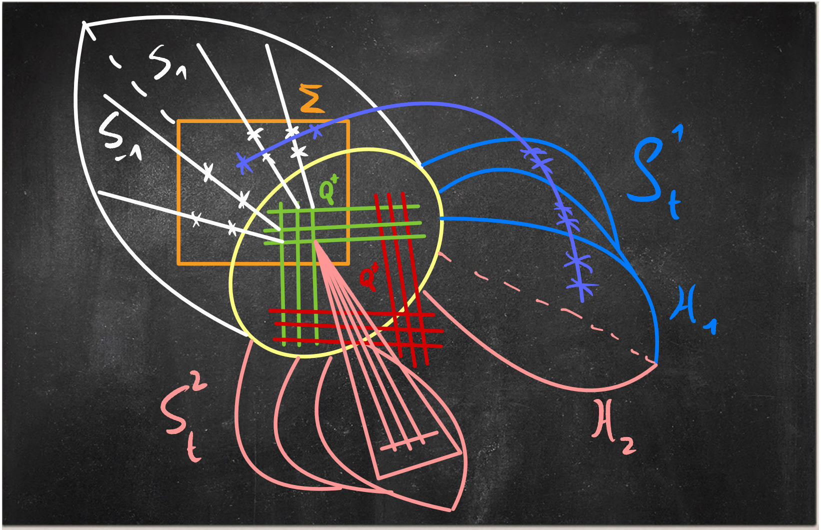
S_1
S_1
Σ
Q^+
Q^+
S_t^1
$\mathcal{H}_1$
S_t^2
$\mathcal{H}_2$

Stabiliser of Q_0

G acts on $\mathcal{H} \setminus \mathcal{Q}$ with two orbits, H_1 and H_2

G also stabilises the following varieties:

$$\mathcal{S}_t := \{\langle v \rangle : H - 2tQ^{\frac{q+1}{2}} = 0\}$$

The surfaces $\mathcal{S}_t$ partition the points outside of $H \cup Q \cup \Sigma$. The points of $\mathcal{S}_t \setminus \mathcal{H} \cap \mathcal{Q}^+$ correspond to non-singular tensors if and only if $1 - 1/t^2 \in \square_q^\times$.

$\implies$ two families of surfaces:

- the non-singular $\mathcal{S}_i^1$, with $1 - 1/i^2 \in \square_q^\times$;
- the singular $\mathcal{S}_j^2$, with $1 - 1/j^2 \notin \square_q^\times$.

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- the singular $\mathcal{S}_j^2$, with $1 - 1/j^2 \notin \square_q^\times$.

Stabiliser of Q_0

G acts on $\mathcal{H} \setminus \mathcal{Q}$ with two orbits, H_1 and H_2
G also stabilises the following varieties:

$$\mathcal{S}_t := \{\langle v \rangle : H - 2tQ^{\frac{q+1}{2}} = 0\}$$

The surfaces $\mathcal{S}_t$ partition the points outside of $H \cup Q \cup \Sigma$. The points of $\mathcal{S}_t \setminus \mathcal{H} \cap \mathcal{Q}^+$ correspond to non-singular tensors if and only if $1 - 1/t^2 \in \square_q^\times$.

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New quasi-Hermitian surfaces from the orbits of G

Theorem (SL-Sheekey)

For any pair of surfaces $\mathcal{S}_i^1, \mathcal{S}_j^2$, their join $\mathcal{S}_i^1, \cup \mathcal{S}_j^2$ is a quasi-hermitian surface, not isomorphic to any previous construction. These are roughly $q^2/4$ new quasi-hermitian surfaces.

Observe that:

- We have a nice equation for $\mathcal{S}_i^1, \cup \mathcal{S}_j^2$, which is

$$(H - 2iQ^{\frac{q+1}{2}})(H - 2jQ^{\frac{q+1}{2}}) = 0$$

- from the equation it seems it is very close to an hermitian surface, we just added some perturbation
- these quasi-hermitian surfaces clearly have a group isomorphic to $\mathrm{PCSO}^+(4, q)$ as automorphism group

...and now?

- The picture still requires more explorations: understanding non-singular sublines in our picture implies some understanding of the *four tensors*, whose contractions are non-singular-sublines. This would have a major impact on semifields.

- Is this a happy island or also the general dimension case is so full of nice geometry? We do believe that the quasi-Hermitian variety will occur in higher dimensions (so new strongly regular graphs).

- There are some more geometrical structures related, among which a partition of the subgeometry in quadrics..the picture is much more rich!