Universality for graphs of bounded degeneracy

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Universality

- H is a subgraph of G if there is an injective $\varphi : V(H) \to V(G)$ such that
  $$uv \in E(H) \quad \Longrightarrow \quad \varphi(u)\varphi(v) \in E(G)$$
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Why?

- For every i = 1,2,... there is a tree with i vertices of degree $\sim$ n/i

- If G universal then its degree sequence is at least $(n, n/2, n/3, \ldots)$
- So $e(G) \geq \dfrac{1}{2} \displaystyle\sum_{i=1}^{n} \frac{n}{i} = \Omega(n \log n)$

What is $\min\{e(G) : G$ is $\mathscr{H}$-universal$\}$?

Chung & Graham 1983

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Alon & Capalbo 2008

- There is a graph with $O(n^{2-2/\Delta})$ edges that is universal for $\mathcal{H} = \{n\text{-vertex graphs of maximum degree } \Delta\}$.

  → best possible order of magnitude

For which p is G(N,p) $\mathscr{H}$-universal?

- For some $p = \Theta(1/n)$, $G(Cn, p)$ is a.a.s. universal for $\mathscr{T}(n, \Delta) := \{n\text{-vertex trees of maximum degree } \Delta\}$. Friedman & Pippenger 1986
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Conlon, Ferber, Nenadov & Škorić 2017

- For some $p = \tilde{\Theta}(n^{-1/(\Delta-1)})$, $G((1+\varepsilon)n, p)$ is a.a.s. $\mathcal{H}_\Delta(n)$-universal.

$\rightarrow$ $p = \Omega(n^{-2/(\Delta+1)})$ is necessary

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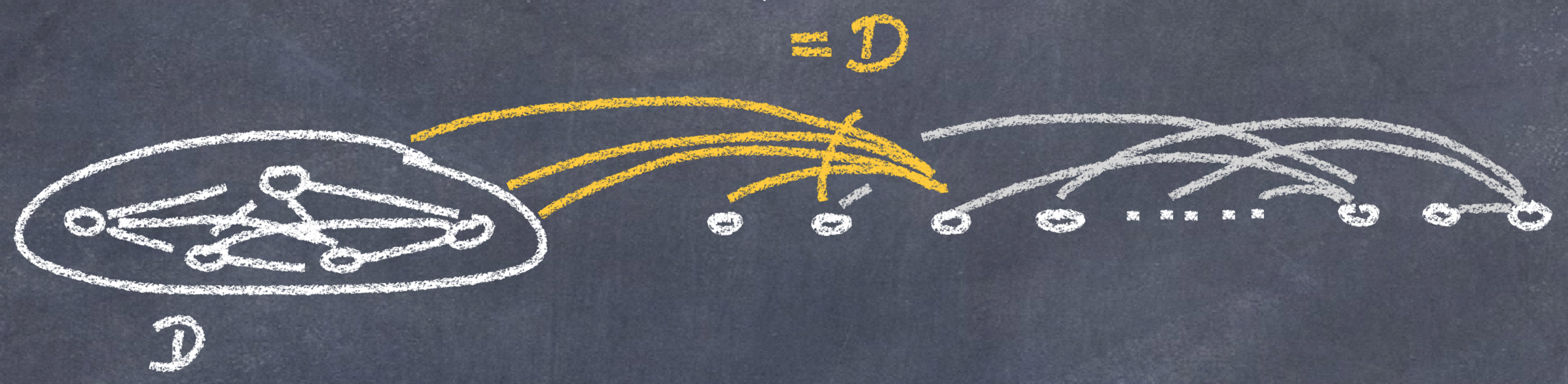
Question (Alon 2019)

What is $\min\{e(G) : G$ is $\mathscr{H}$-universal$\}$ for $\mathscr{H} = \mathscr{H}(n, D)$?

A counting lower bound

- Suppose G is $\mathscr{H}(n, D)$-universal.
- Count "full" D-degenerate graphs (in order) on [n]:



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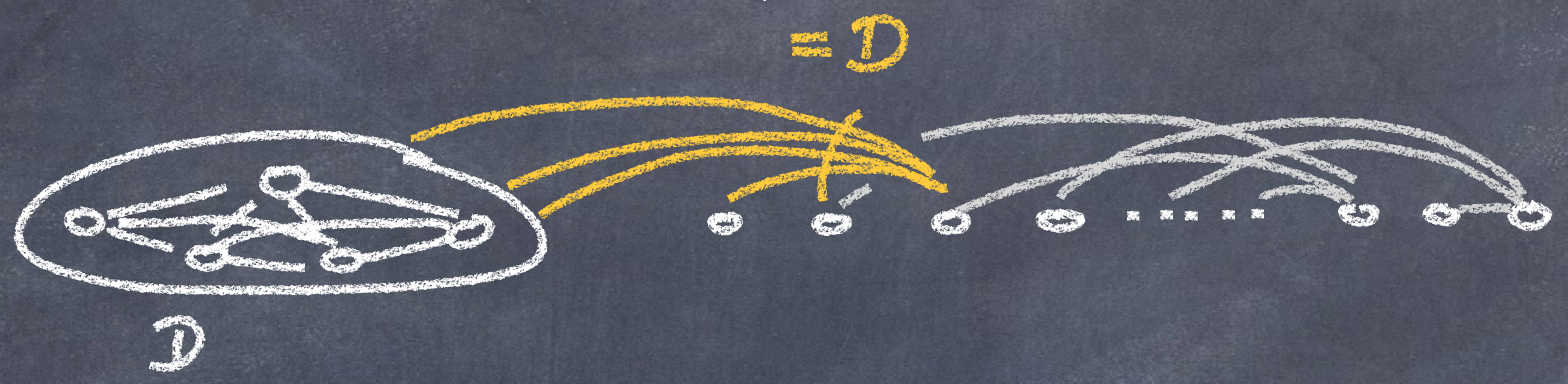

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- How many are there in G?
  - Pick any Dn edges and any ordering of the n vertices they span.
  - at most $\binom{e(G)}{Dn} n! \leq \left(\frac{e(G) \cdot e}{Dn}\right)^{Dn} n^n$

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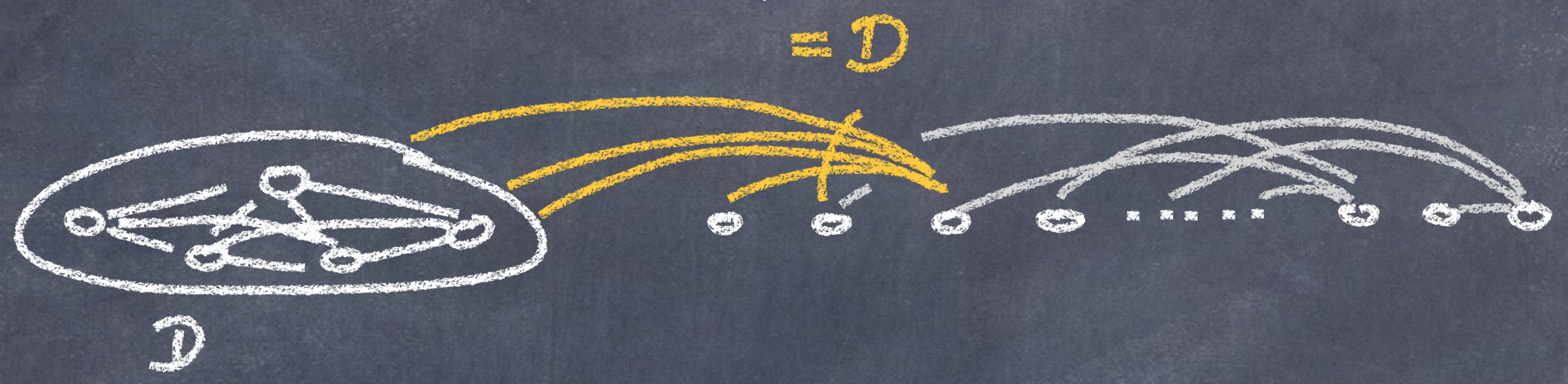

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- So : $e(G) \geq cn^{2-1/D}$

Universality for $\mathscr{H}(n, D)$

Theorem (Allen, Böttcher, L. 2023+)

There exists a graph G with $n^{2-1/D}\text{polylog}(n)$ edges that is $\mathscr{H}(n, D)$-universal.

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Observation 2: $e(H) \leq Dn$

- So # { vertices of degree $\geq k$ } $\leq 2Dn/k$.
- In particular, # { vertices of degree $\geq \varepsilon n$ } $\leq 2D\varepsilon^{-1}$.

Construction: Random block model

Remember: # { vertices in H of degree $\geq$ k } $\leq$ 2Dn/k

G

embed vertices of degree $\geq n^{1/D}$

$|W_1| \approx n^{1-1/D}$

W_2

. . .

$|W_i| \approx n^{1-1/D^i}$

$p_i \approx n^{\frac{2}{D^i}-\frac{1}{D}}$

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edge probability

$p_{i,j} \approx n^{-\frac{1}{D}+\frac{1}{D^i}+\frac{1}{D^j}}$

▶ $\mathbb{E}\, e(W_i, W_j) \approx n^{2-1/D}$

$|W_j| \approx n^{1-1/D^j}$

$|W_M| \approx n$ $(M \sim \log\log n)$

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embed vertices of degree $\leq \Delta = 3^D$

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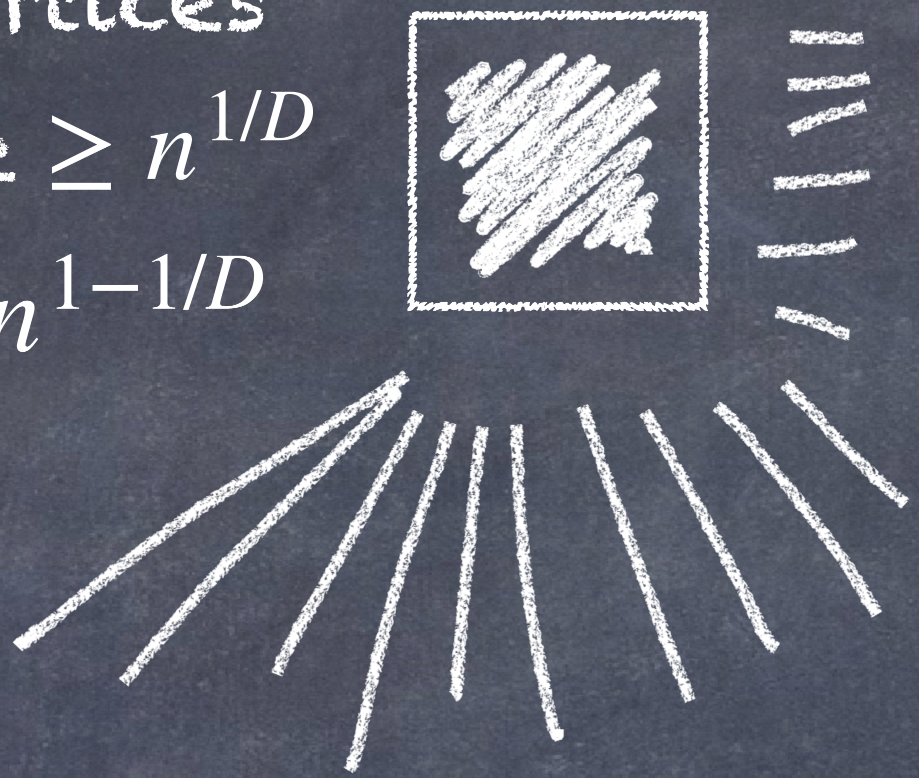

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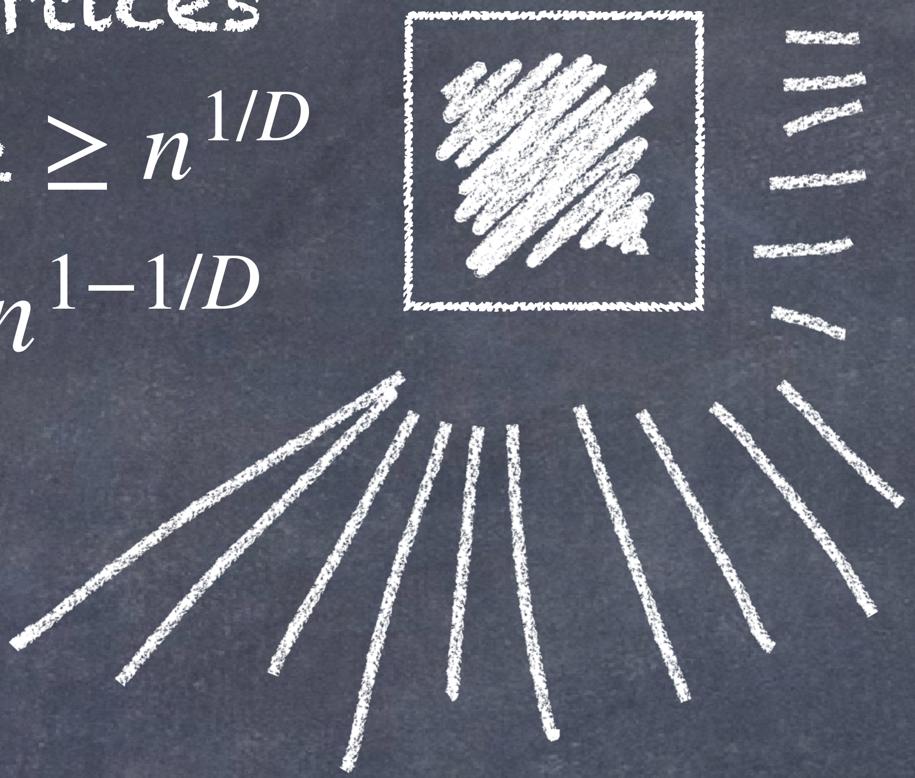

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▷ $G(Cn, p_M)$ is $\mathcal{H}_\Delta(n, D)$-universal

(Nenadov 2016)

Open problems

- We proved: $n^{2-1/D}\mathrm{polylog}(n)$ edges are sufficient to find an $\mathcal{H}(n,D)$-universal graph G.
- This G has Cn vertices.
  Construction gives (with extra work) C = $(1+\varepsilon)n$.

- Can we find a construction where G has n vertices?

- Can we remove the polylog(n) factor?

- Almost the same bound as for $\mathcal{H}_\Delta(n,D)$-universality.

- When is $G(n,p)$ universal for $\mathcal{H}_\Delta(n)$?

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