Hello and Welcome

Generalisations of the Erdős-Ko-Rado Theorem for Permutations

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The Erdős-Ko-Rado Theorem

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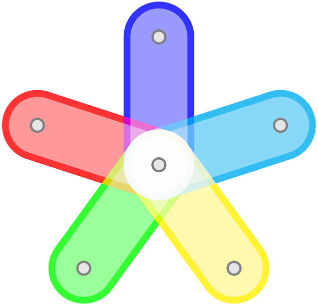

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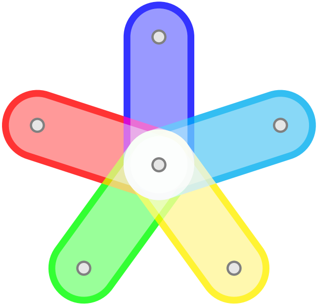

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If $\mathscr{F} \subseteq \binom{[n]}{k}$ *is intersecting and* $n \geq 2k$*, then*

$$|\mathscr{F}| \leq \binom{n-1}{k-1}$$

with equality for $n > 2k$ *if and only if* $\mathscr{F}$ *is a star.*

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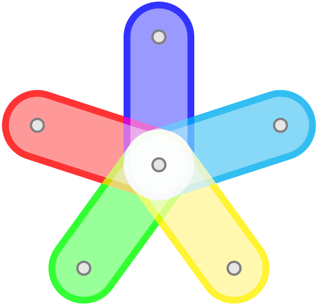

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Proof.

Let $\sigma = (1\ 2 \cdots n)$ and $H = \langle \sigma \rangle$. The elements $\tau\sigma^i$ and $\tau\sigma^j$, intersect if and only if $\sigma^{-i}\sigma^j$ has a fixed point. So no two elements of the same coset of H of S_n can be in $\mathscr{F}$. There are $(n-1)!$ cosets. $\square$

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Theorem (Cameron & Ku 2003, Larose & Malvenuto 2004)

Equality holds if and only if $\mathscr{F}$ is a star.

Permutations as Matchings of $K_{n,n}$

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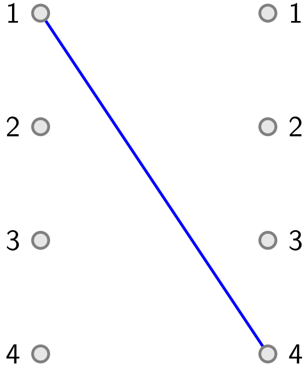

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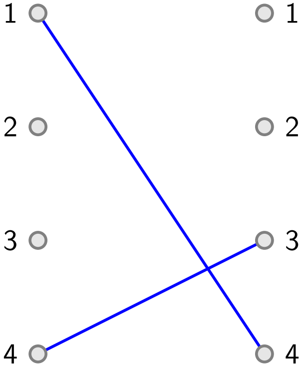

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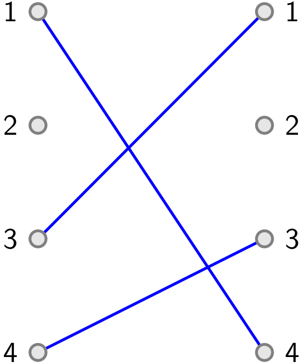

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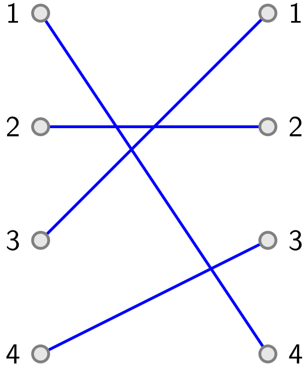

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- $r < \min\{n, m\}$ Borg and Meagher (2015)

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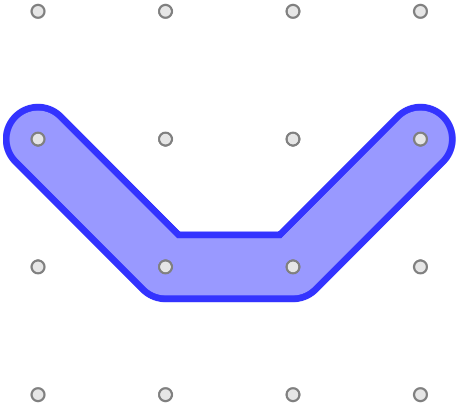

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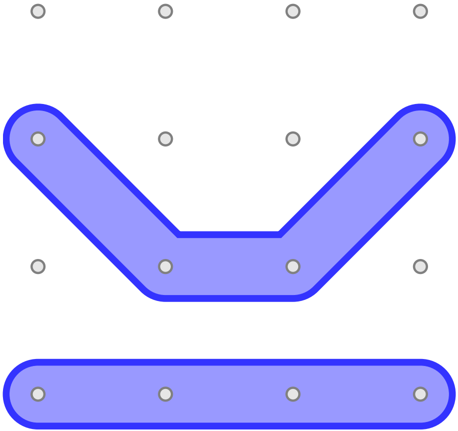

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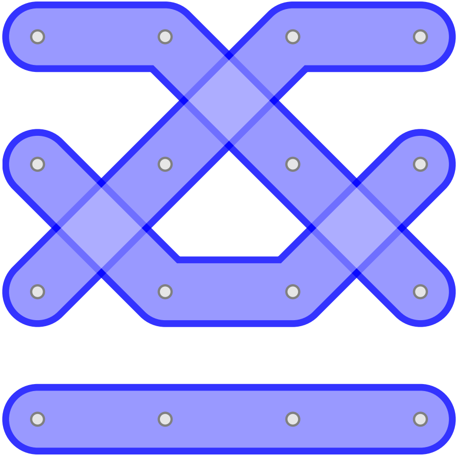

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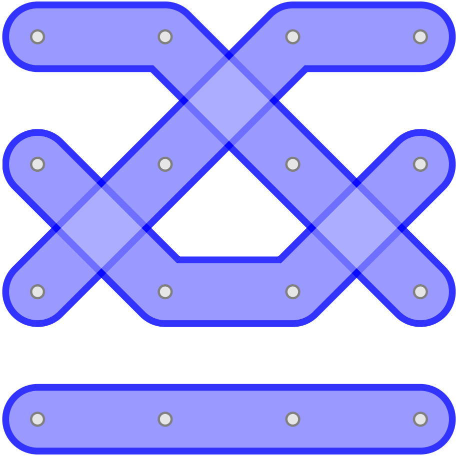
Matching of $\mathcal{K}_{4,4,4,4}$



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- Let $\mathcal{M}_r(n_1, \ldots, n_k)$ be the set of matchings of size r of $\mathcal{K}_{n_1,\ldots,n_k}$.

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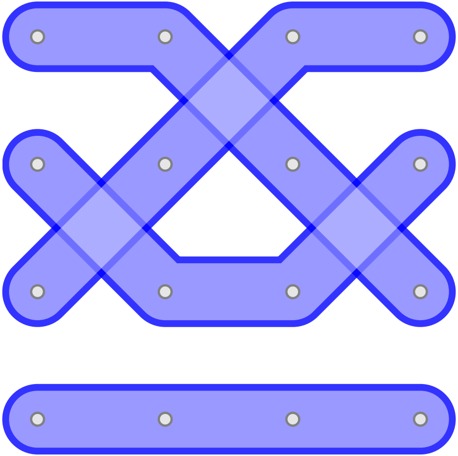

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- When $k=1$ we get the Erdős-Ko-Rado Theorem for $n>2r$.

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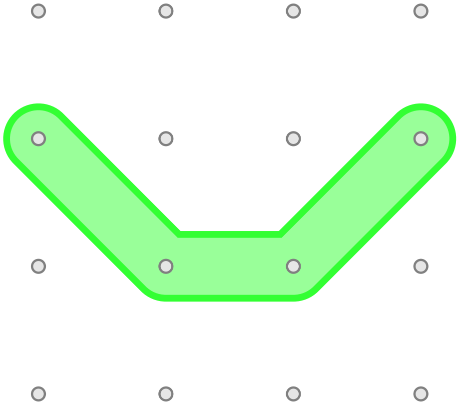

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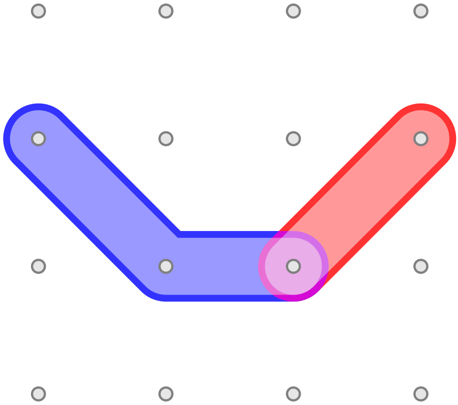

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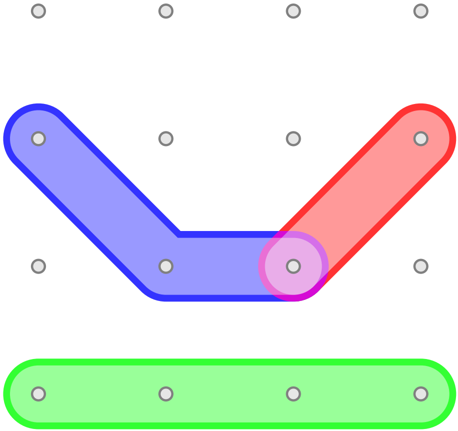

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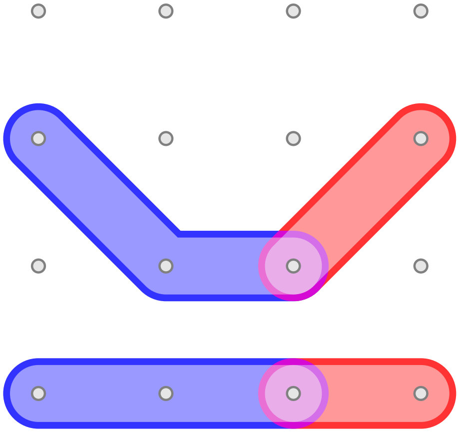

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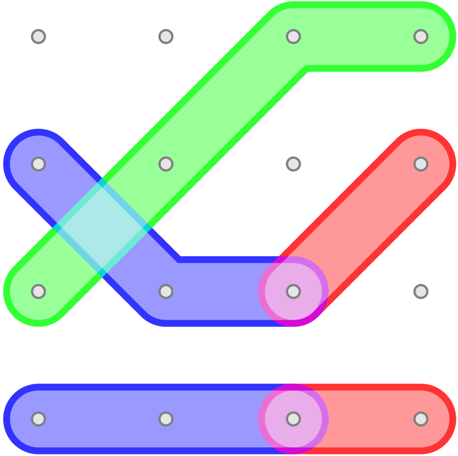

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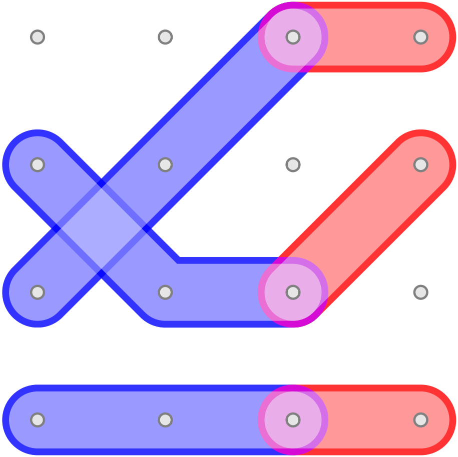

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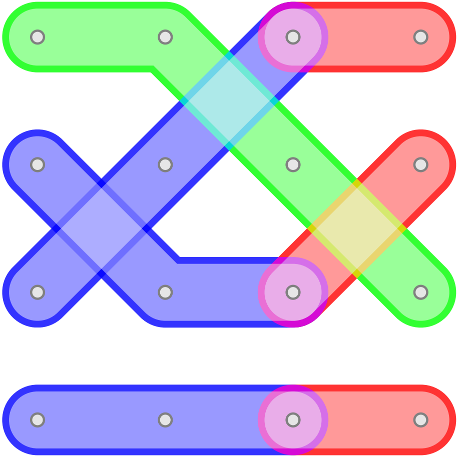

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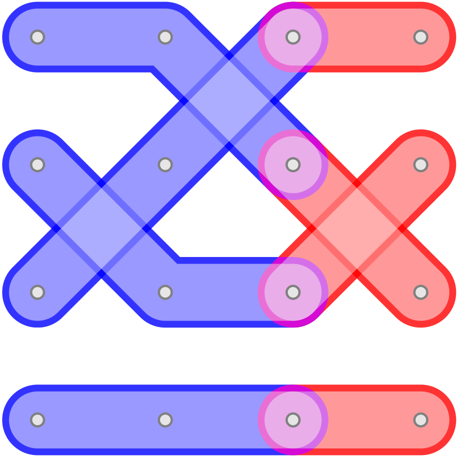

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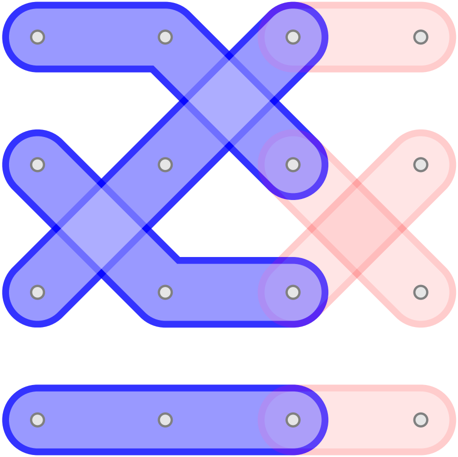

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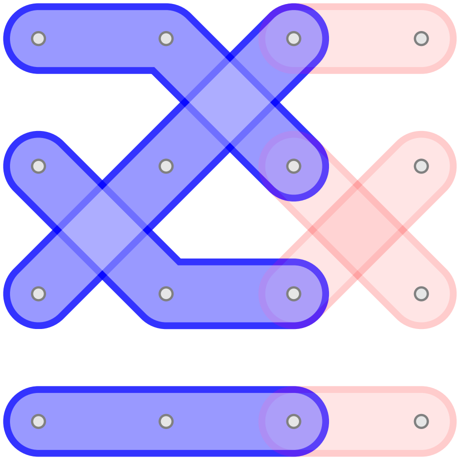

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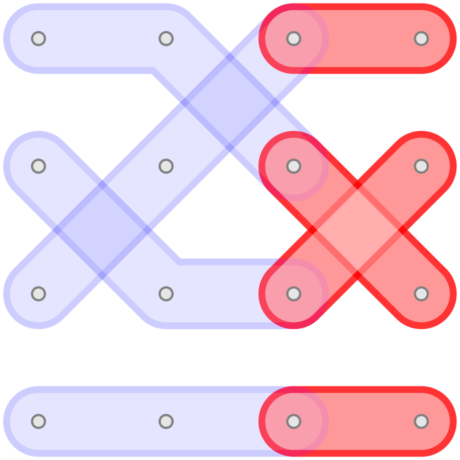

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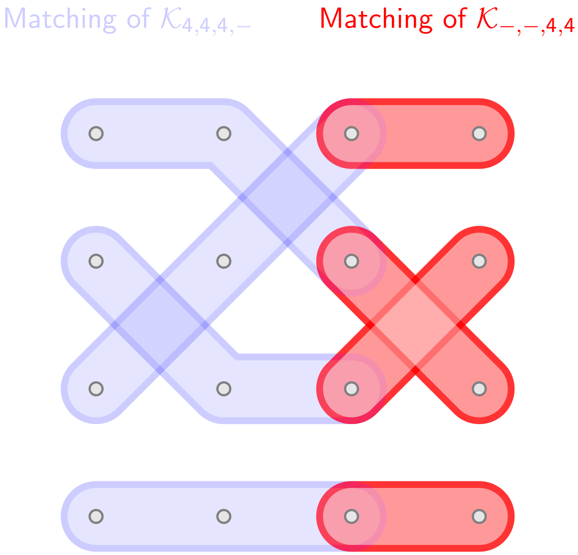

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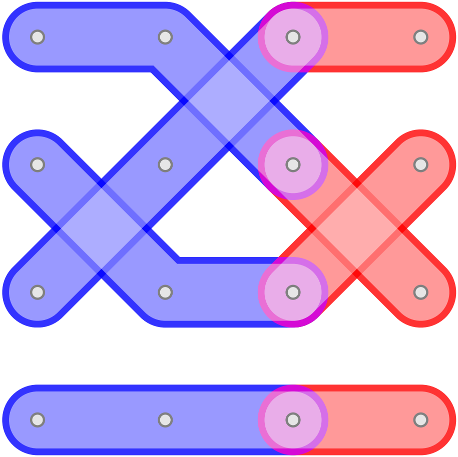

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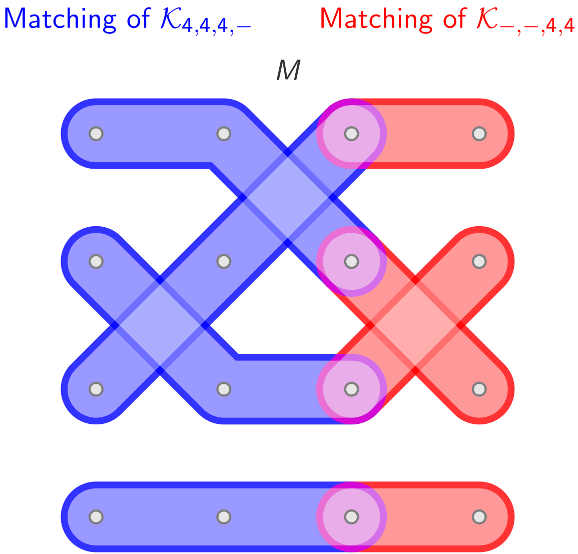

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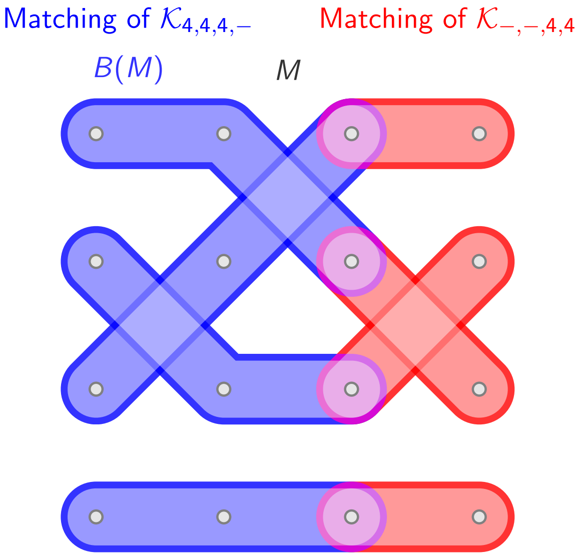

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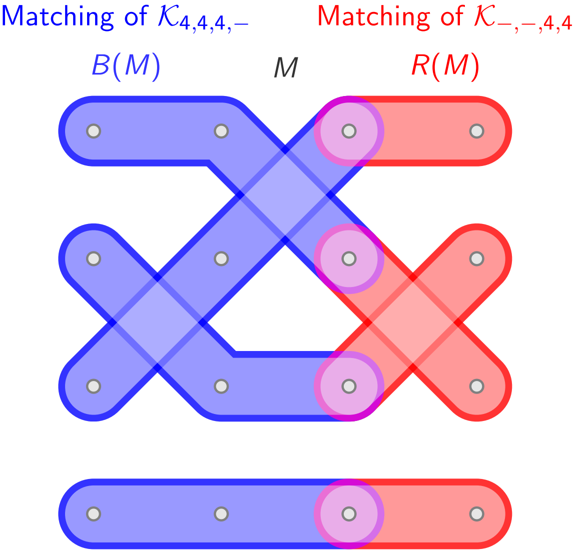

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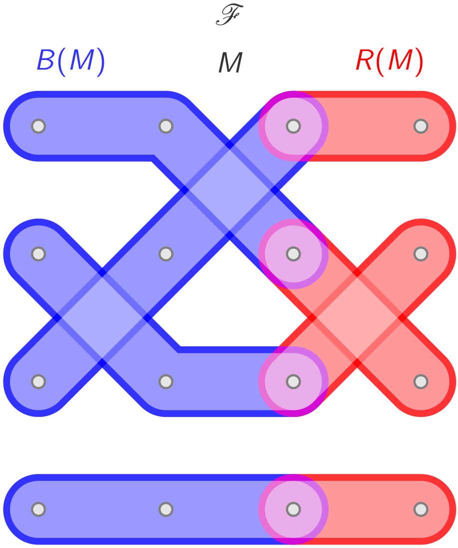

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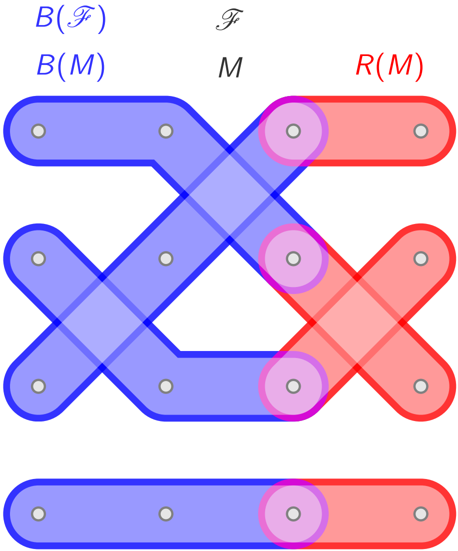

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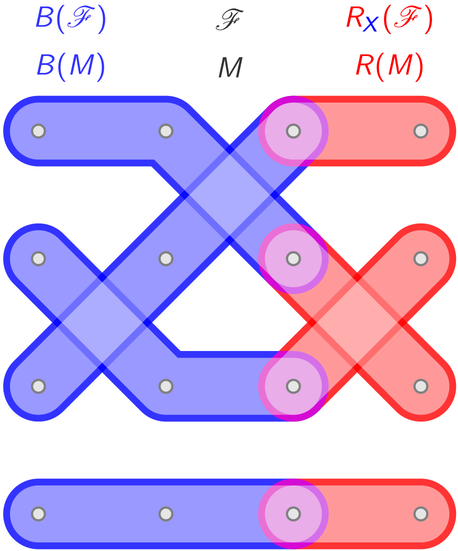

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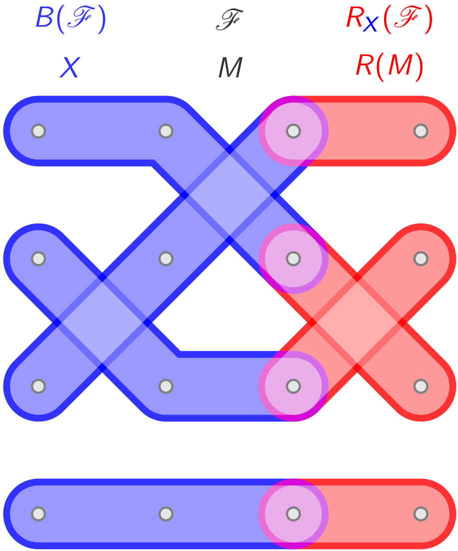

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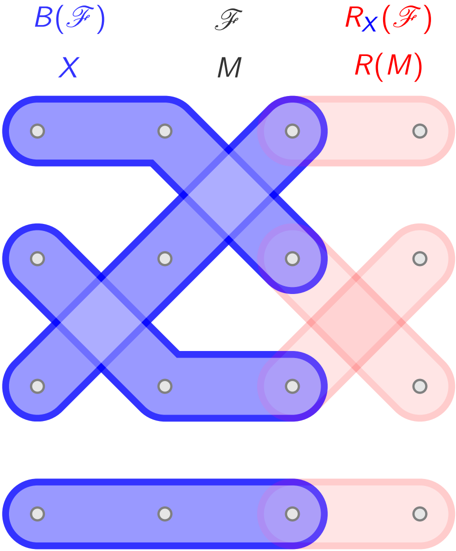

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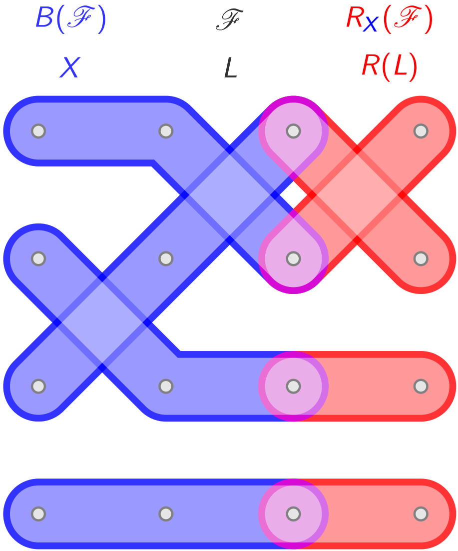

Proof Sketch

Theorem (M.)

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1, \ldots, n_k)$ be an intersecting family. Then

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(ii) $|\mathscr{F}| = \sum_{X \in B(\mathscr{F})} |R_X(\mathscr{F})|$

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Theorem (M.)

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1, \ldots, n_k)$ be an intersecting family. Then

$$|\mathscr{F}| \leq \frac{(n_1 - 1)_{r-1} \cdots (n_k - 1)_{r-1}}{(r-1)_{r-1}}.$$

Furthermore, equality holds if and only if $\mathscr{F}$ is a star.

Lemma

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1, \ldots, n_k)$ be an intersecting family. Then

(i) $B(\mathscr{F})$ *and* $R_X(\mathscr{F})$ *are intersecting*

(ii) $|\mathscr{F}| = \sum_{X \in B(\mathscr{F})} |R_X(\mathscr{F})|$

(iii) *if* $B(\mathscr{F})$ *and all* $R_X(\mathscr{F})$ *are stars, then* $\mathscr{F}$ *is a star.*

Proof Sketch

Proof Sketch.

Lemma

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1, \ldots, n_k)$ *be an intersecting family. Then*

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Proof Sketch

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$$
\begin{aligned}
|\mathscr{F}| &= \sum_{X \in B(\mathscr{F})} |R_X(\mathscr{F})| \\
&\leq \frac{(n_1 - 1)_{r-1} \cdots (n_{k-1} - 1)_{r-1}}{(r-1)_{r-1}} \frac{(r-1)_{r-1}(n_k - 1)_{r-1}}{(r-1)_{r-1}}
\end{aligned}
$$

Lemma

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Proof Sketch

Proof Sketch.

$$\begin{aligned}|\mathscr{F}| &= \sum_{X\in B(\mathscr{F})} |R_X(\mathscr{F})| \\ &\le \frac{(n_1-1)_{r-1}\cdots(n_{k-1}-1)_{r-1}}{(r-1)_{r-1}} \frac{(r-1)_{r-1}(n_k-1)_{r-1}}{(r-1)_{r-1}} \\ &= \frac{(n_1-1)_{r-1}\cdots(n_k-1)_{r-1}}{(r-1)_{r-1}}.\end{aligned}$$

Lemma

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1,\ldots,n_k)$ be an intersecting family. Then

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&= \frac{(n_1-1)_{r-1}\cdots(n_k-1)_{r-1}}{(r-1)_{r-1}}.
\end{aligned}
$$

□

Lemma

Let $\mathscr{F} \subseteq \mathcal{M}_r(n_1, \ldots, n_k)$ be an intersecting family. Then

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The End

Thanks for listening!

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