Automorphisms of direct products of circulant graphs

Đorđe Mitrović

University of Auckland

Supervisors: Gabriel Verret, Jeroen Schillewaert, Florian Lehner

Joint work with Ademir Hujdurović (University of Primorska) and Dave Witte Morris (University of Lethbridge)

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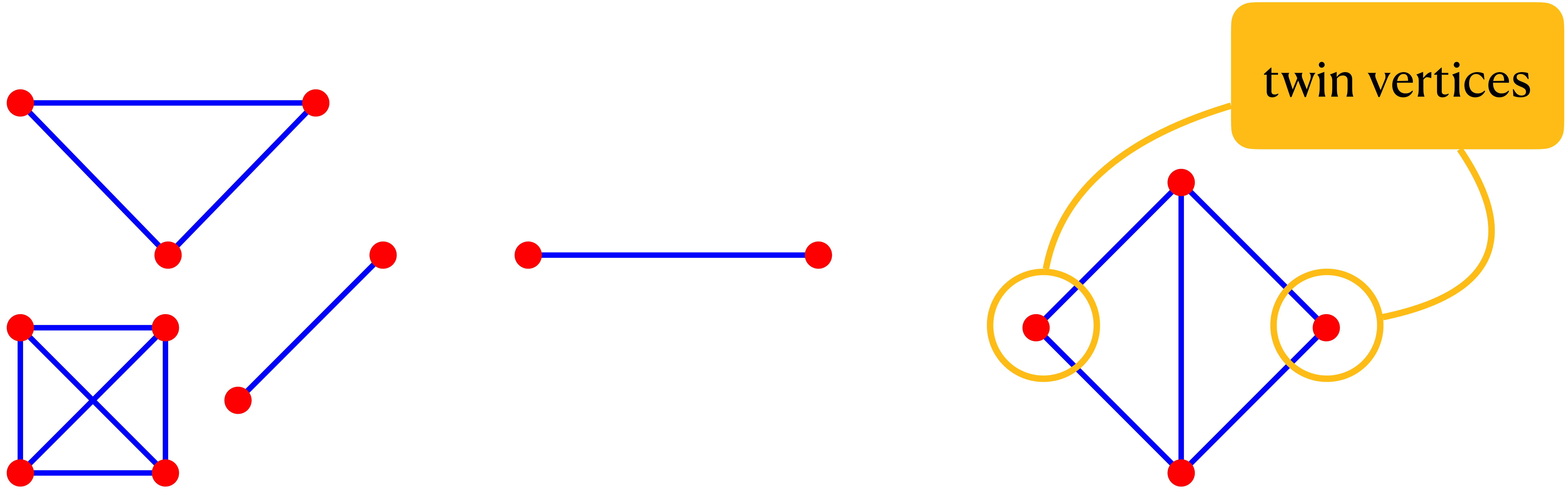
All graphs are **finite**, **simple** and **undirected**.

An **automorphism** of a graph $X = (V, E)$ is a permutation of V that preserves E.

Aut(X) is the **automorphism group** of a graph X.

Twins have the same neighbours.

A graph is **twin-free** if it does not contain any twins.

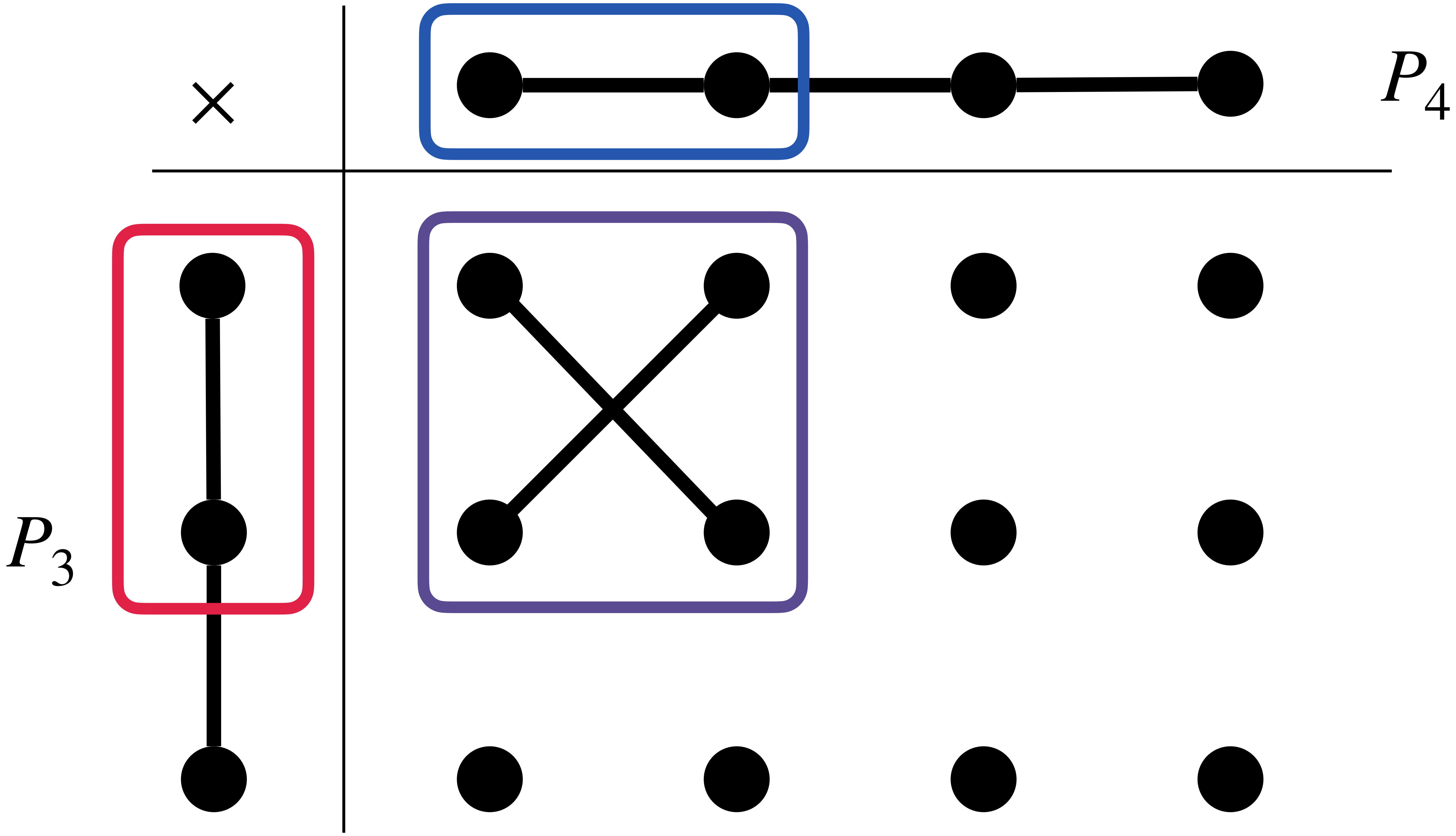


X K_2 $\mathcal{D}$

Direct products of graphs

$$V(X \times Y) = V(X) \times V(Y)$$

$$(x_1, y_1) \sim_{X \times Y} (x_2, y_2) \iff x_1 \sim_X x_2 \text{ and } y_1 \sim_Y y_2$$



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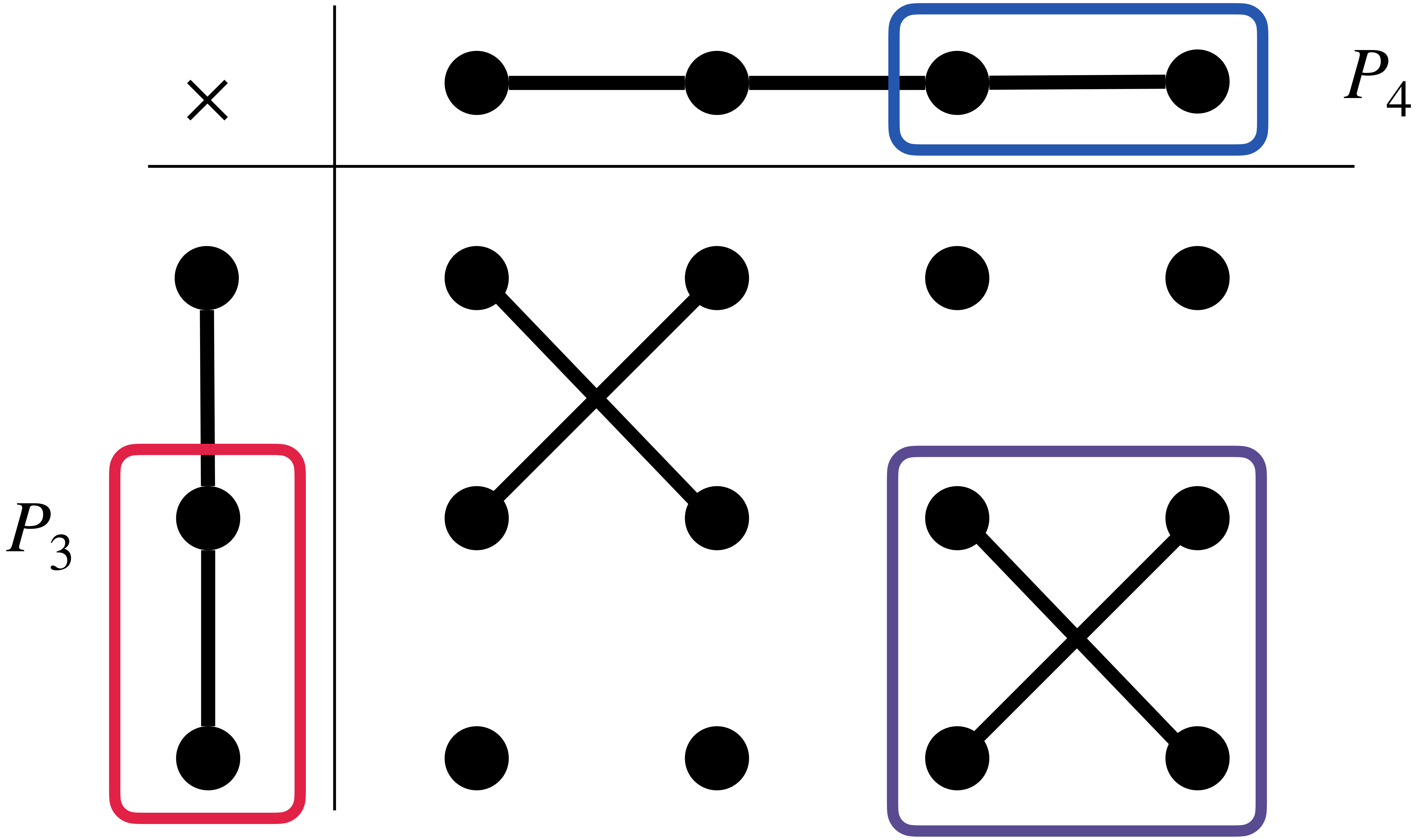

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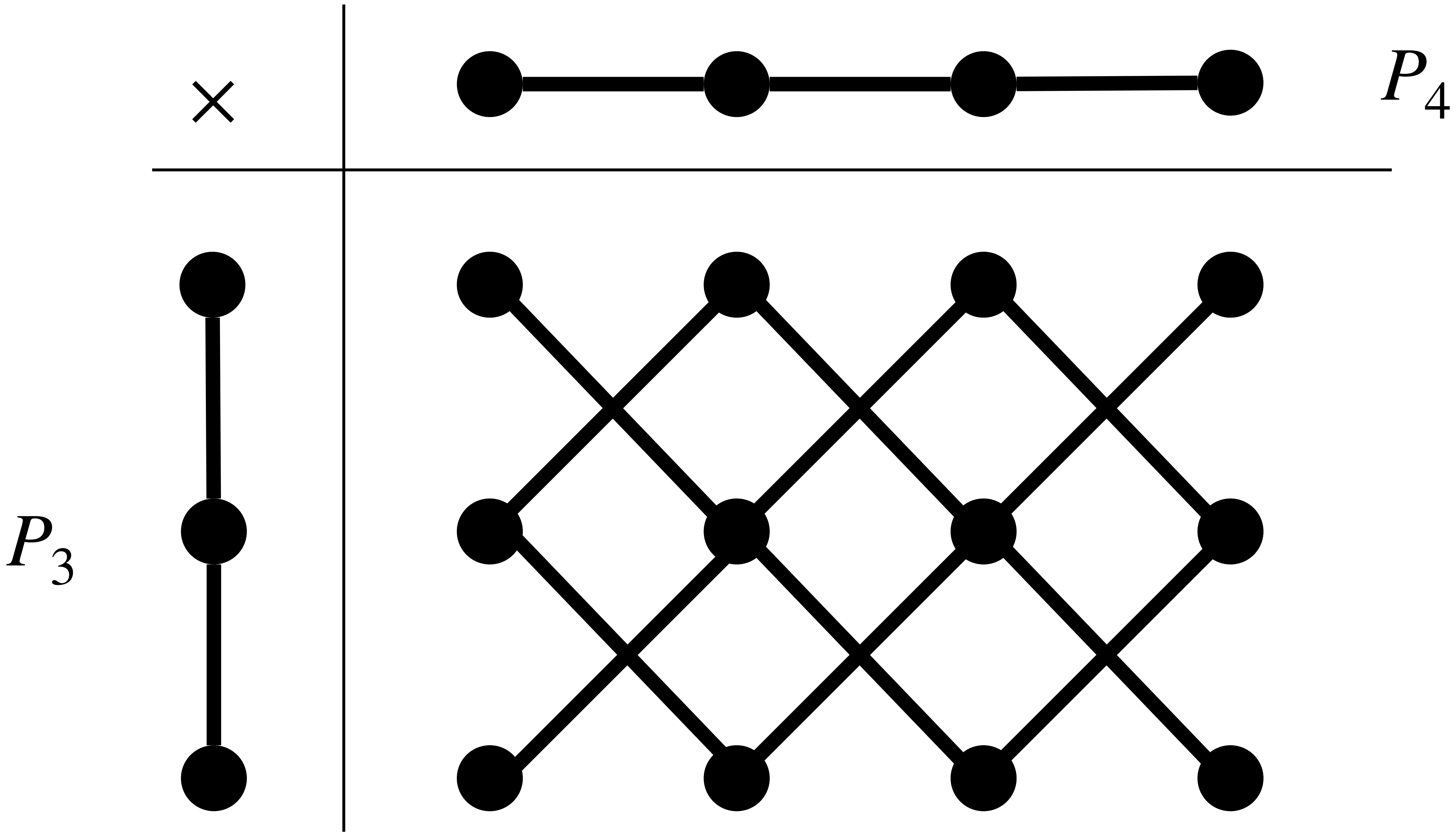

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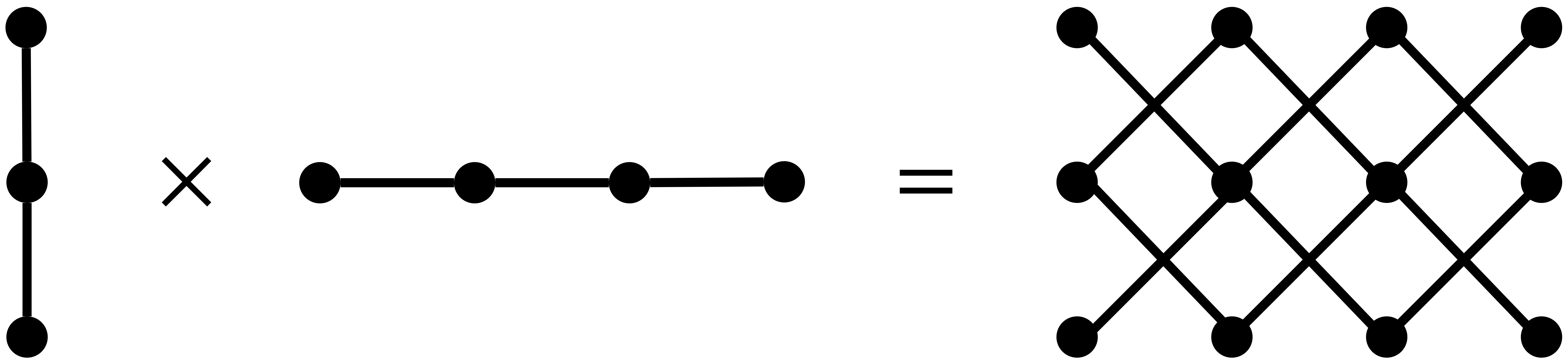
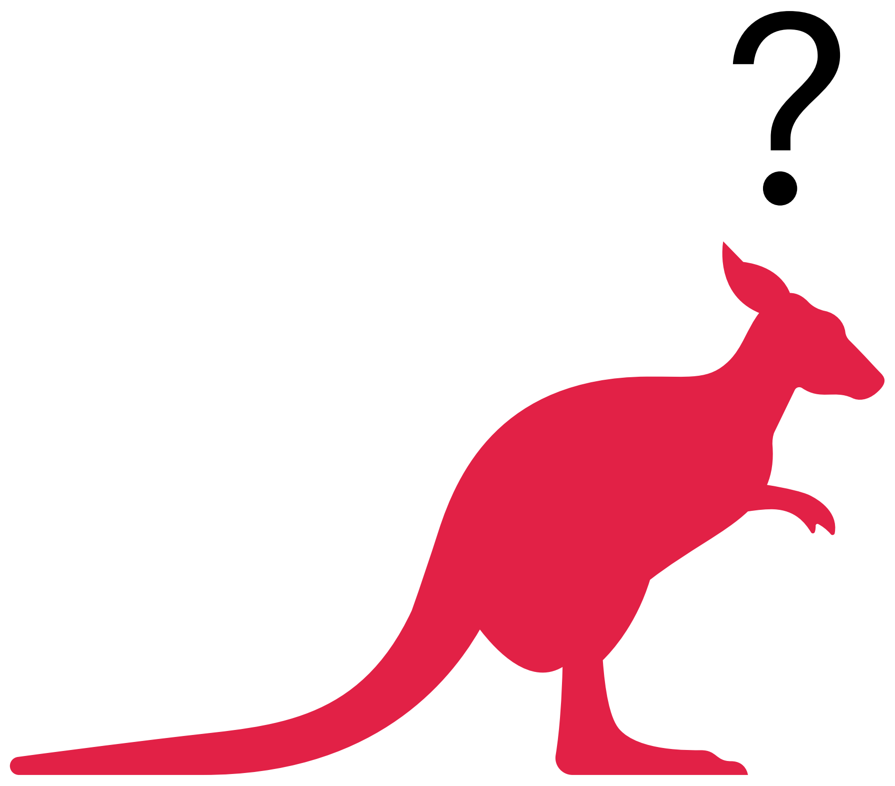
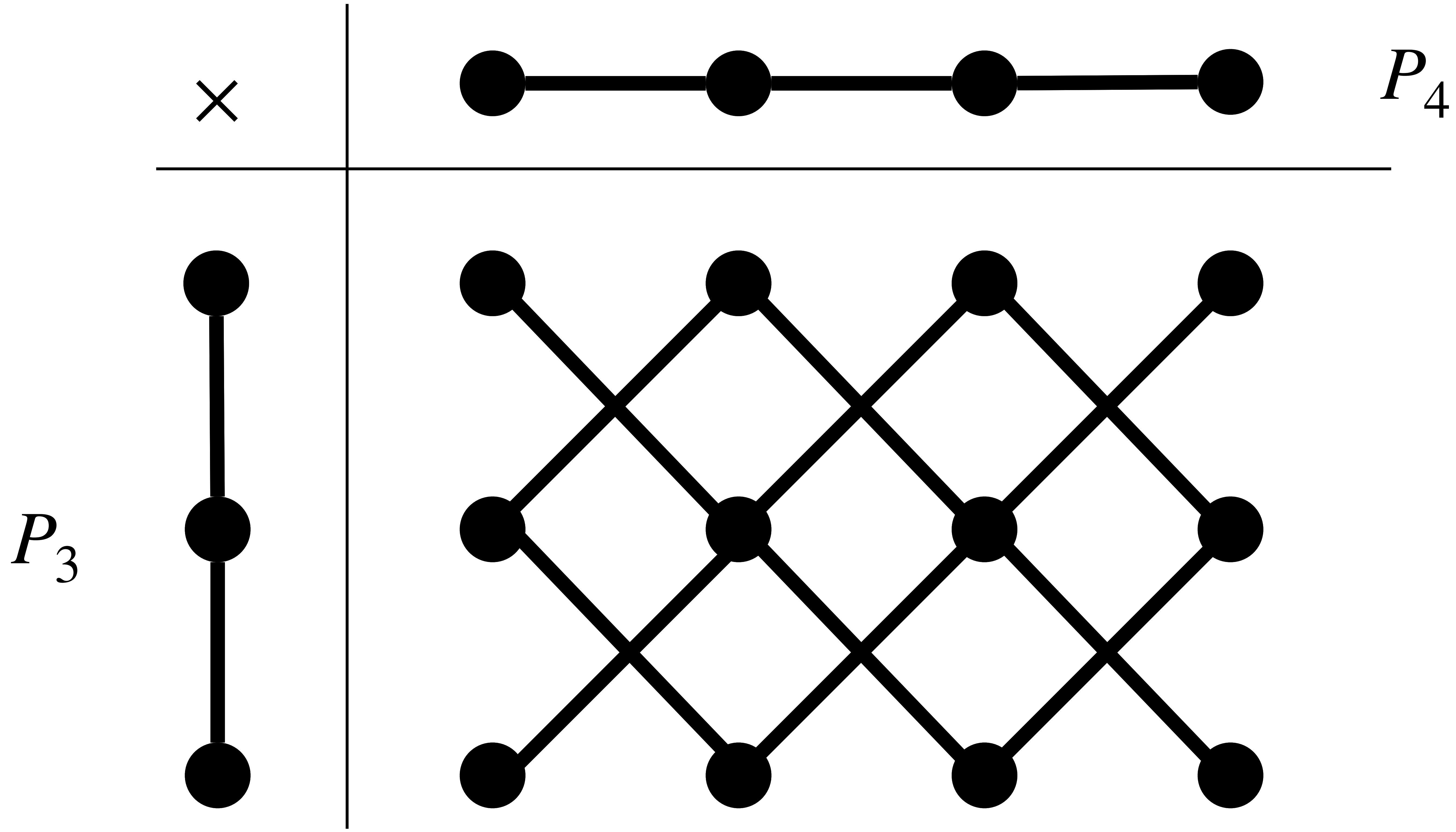
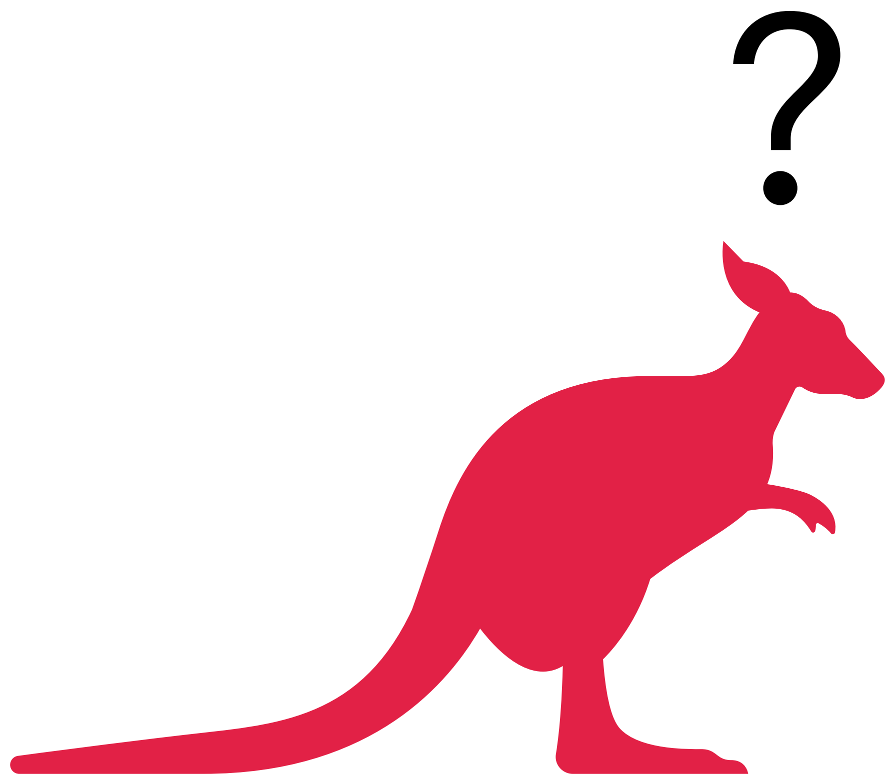




What is $\mathrm{Aut}(X \times Y)$?

$$\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \leq \mathrm{Aut}(X \times Y)$$





When does $\mathrm{Aut}(X) \times \mathrm{Aut}(Y) = \mathrm{Aut}(X \times Y)$?

(What else can $\mathrm{Aut}(X \times Y)$ contain?)

Dörfler's theorem (1974)

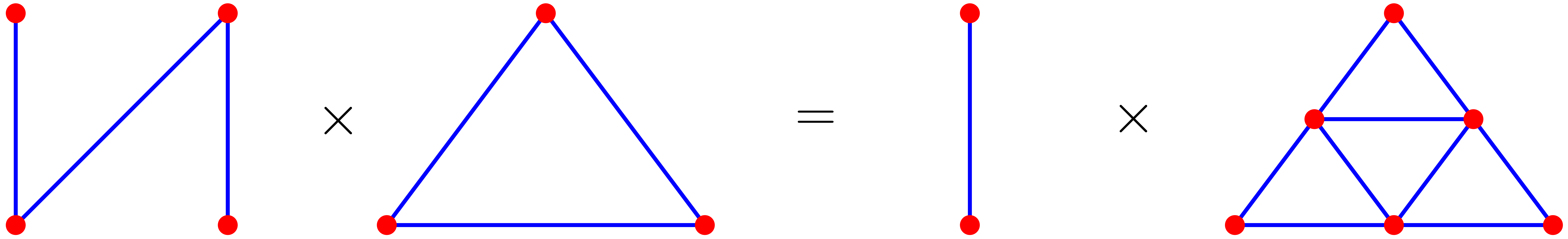
(a complete answer when both graphs are non-bipartite)

Let X and Y be connected, non-bipartite, twin-free graphs with unique prime decompositions $X = X_1 \times \ldots \times X_n$ and $Y = Y_1 \times \ldots \times Y_m$. Then $\mathrm{Aut}(X \times Y)$ is generated by

- automorphisms of X,
- automorphisms of Y,
- permutations of isomorphic factors $X_i \cong Y_j$.

$\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \leq \mathrm{Aut}(X \times Y)$

Failure of uniqueness of the prime factorization wrt $\times$ for bipartite graphs



When is $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$?

(when X is non-bipartite and Y is bipartite)

Reduction to the case $Y = K_2$

Let X be a connected, non-bipartite, twin-free graph such that

$$\mathrm{Aut}(X \times K_2) = \mathrm{Aut}(X) \times \mathrm{Aut}(K_2)$$

Then for every connected, bipartite, twin-free graph Y

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$$

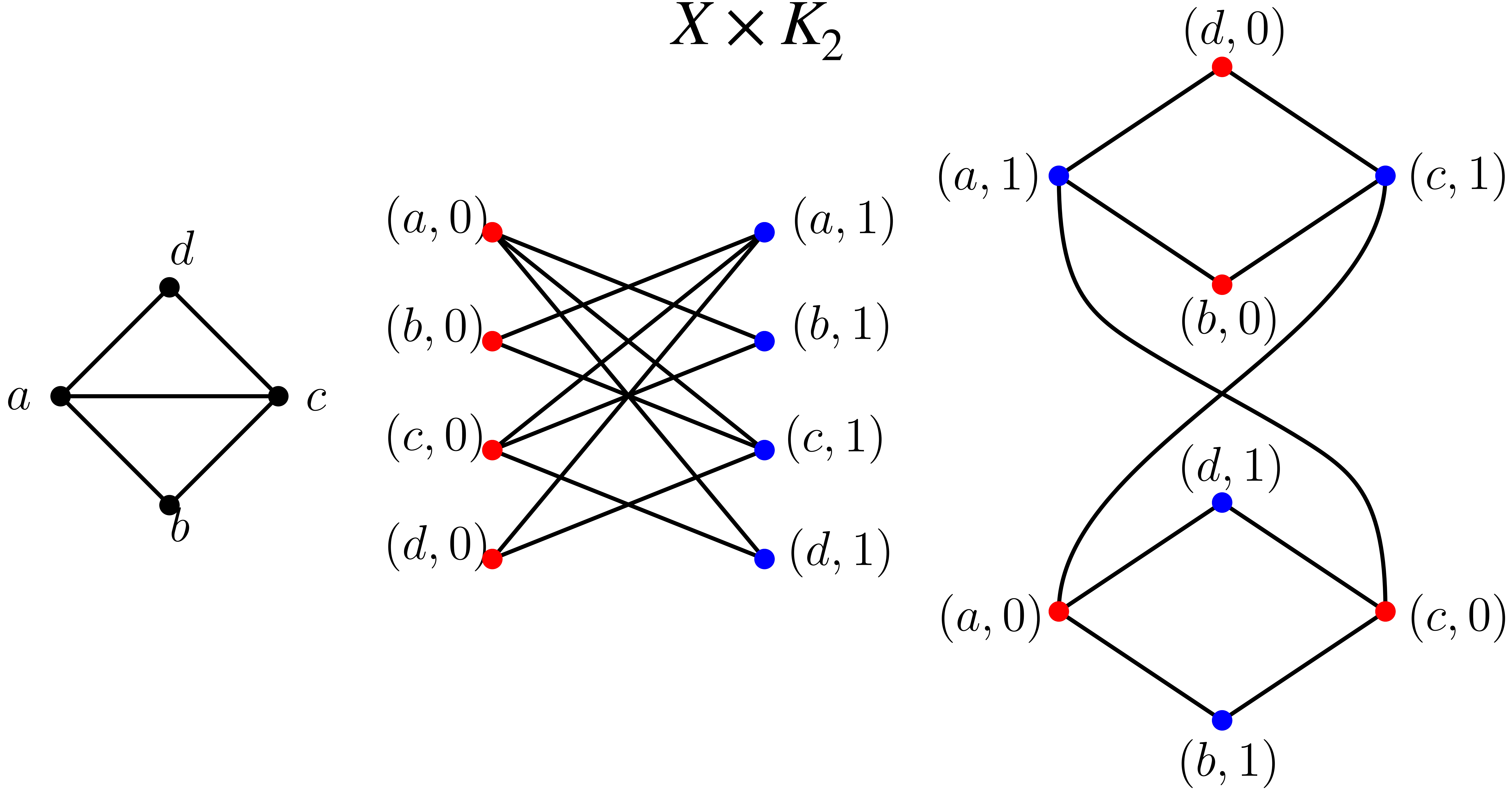
(a folklor result)

+ some mild

technical conditions

Canonical bipartite double cover

$$X \times K_2$$



Main observation

$$\mathrm{Aut}(X) \times \mathrm{Aut}(K_2) \leq \mathrm{Aut}(X \times K_2)$$

Main issue

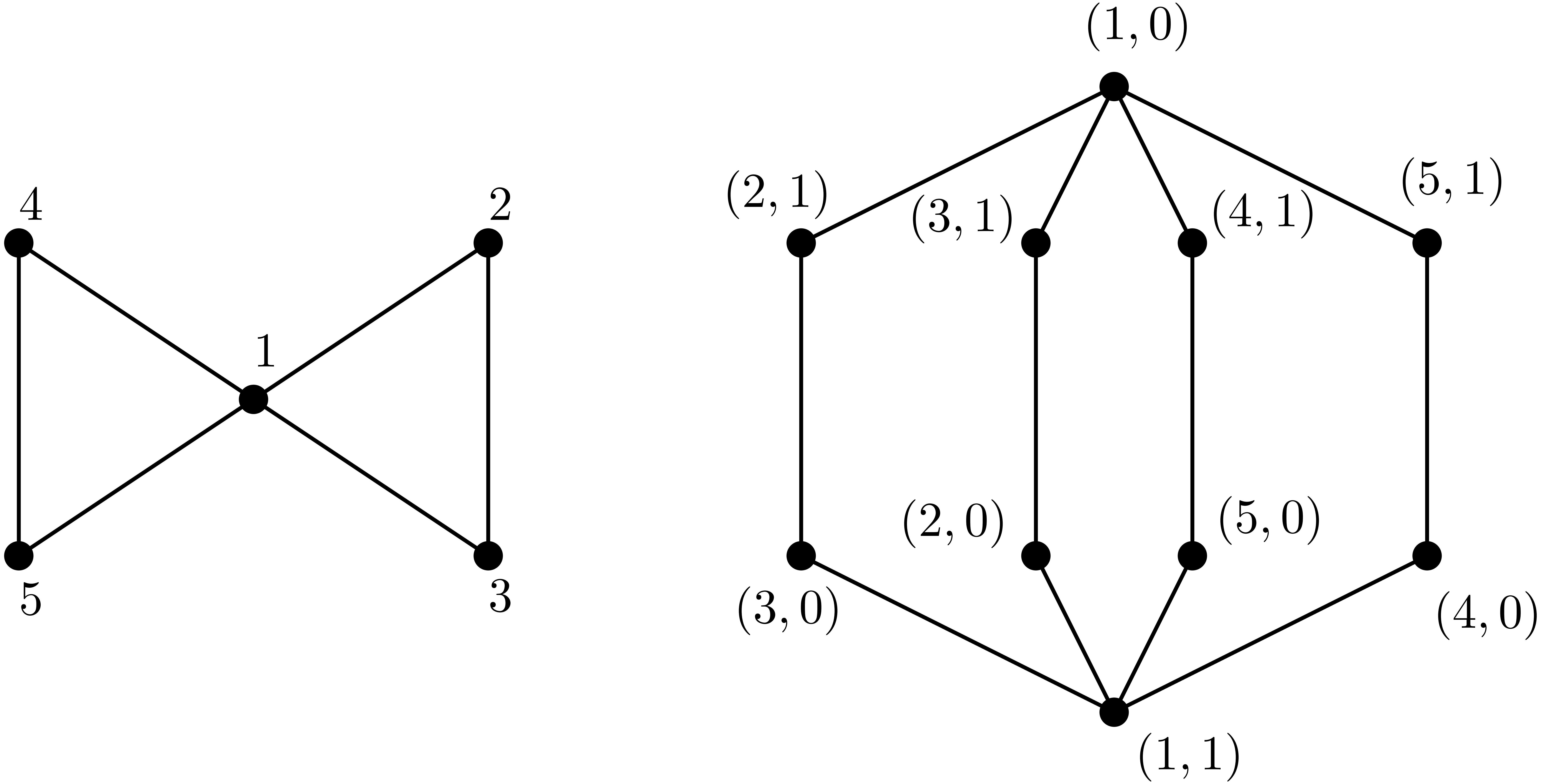
Equality does not always hold!!!

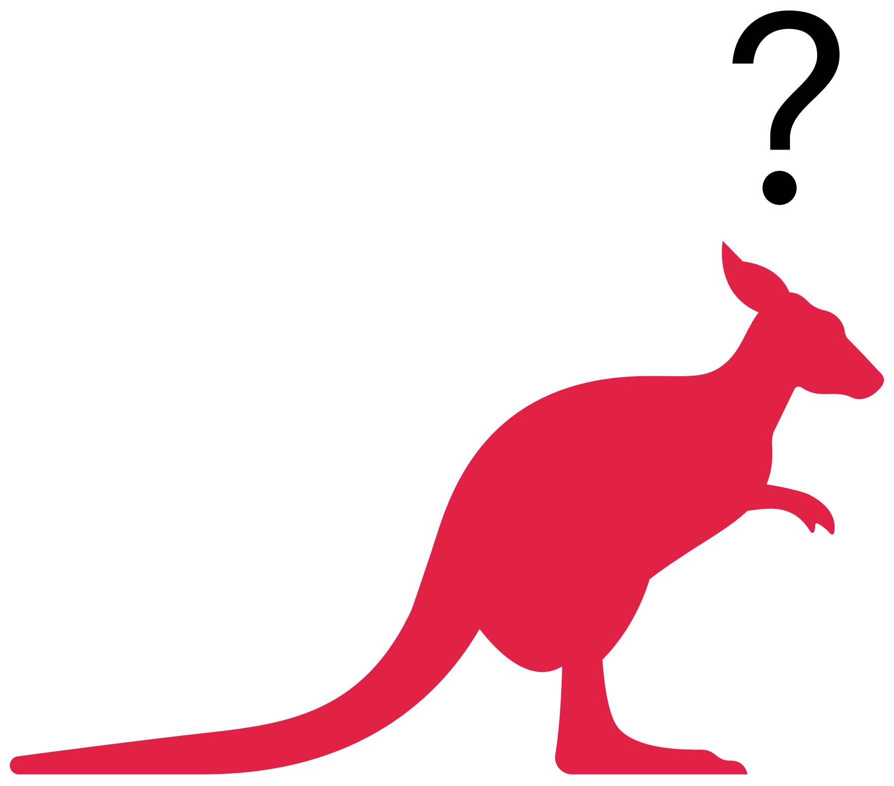
A graph X is called unstable if

$$\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}(X) \times \mathrm{Aut}(K_2)$$

A graph X is called non-trivially unstable if it is

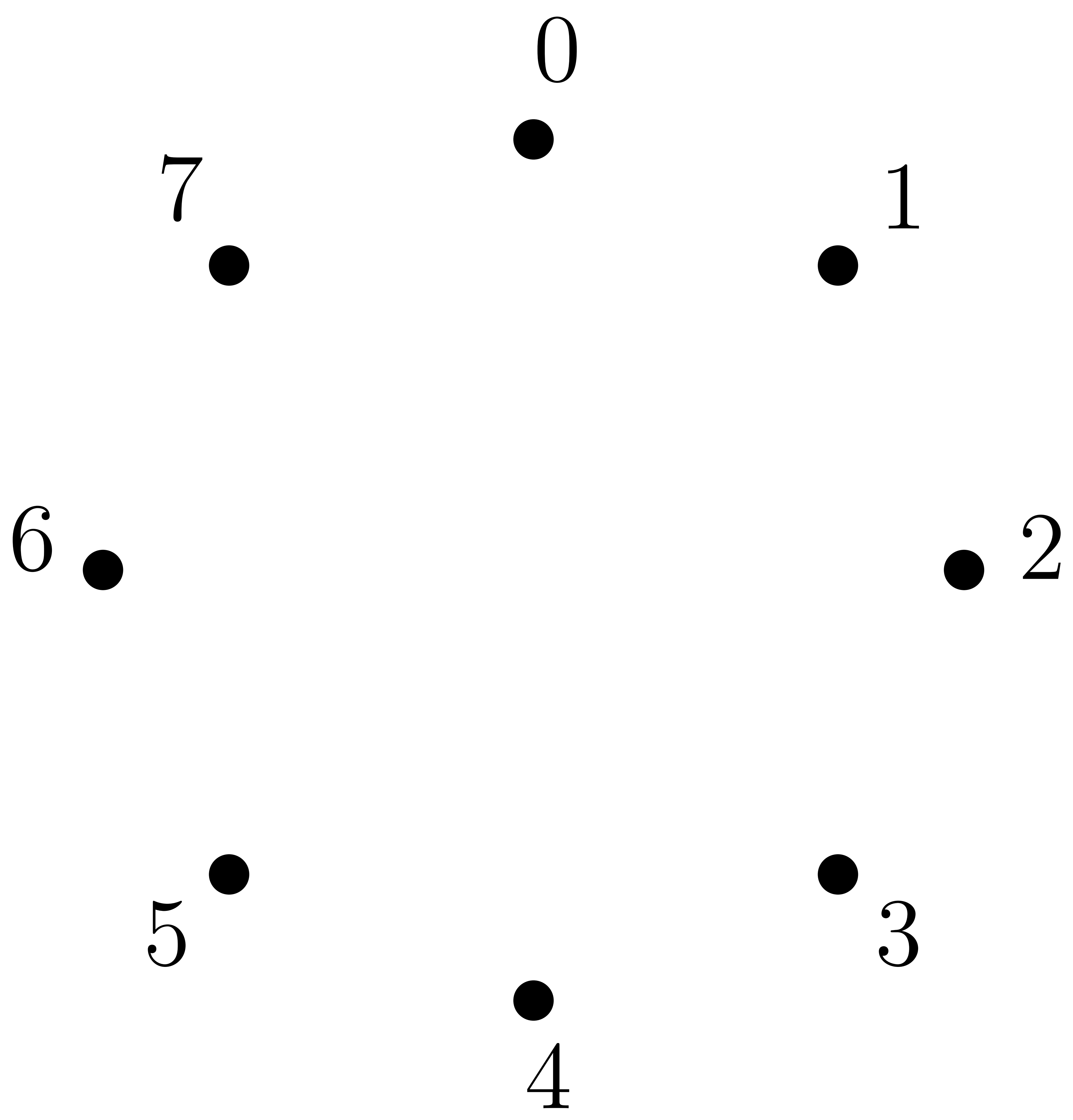
1. connected, non-bipartite, twin-free, and
2. $\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}(X) \times \mathrm{Aut}(K_2)$





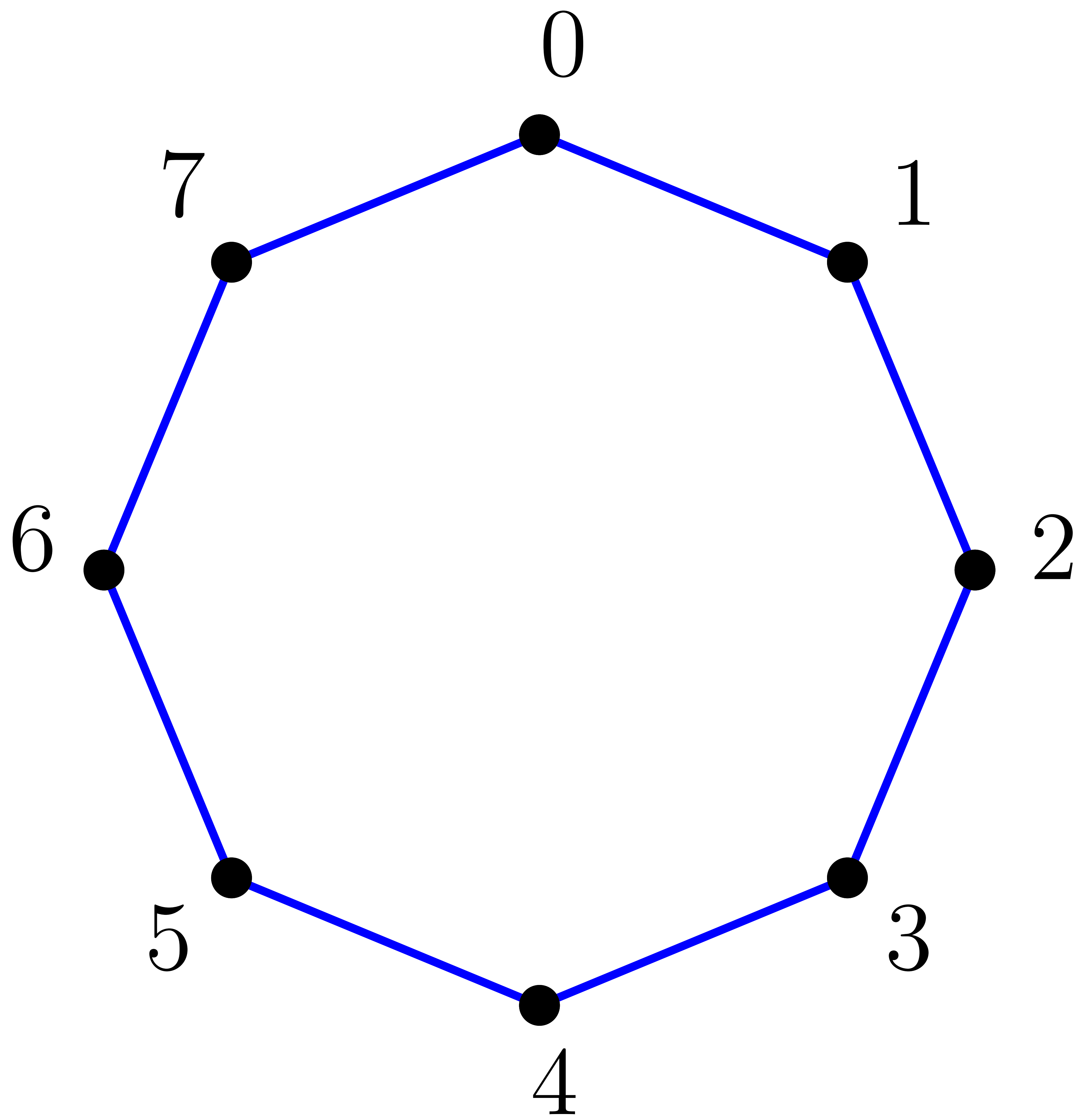
Which circulant graphs are non-trivially unstable? (Wilson 2008)

A circulant graph $\mathrm{Circ}(n, S)$ is a Cayley graph of the cyclic group $\mathbb{Z}_n$.



$\mathrm{Circ}(8, \{\pm 1, \pm 2\})$

A circulant graph Circ(n, S) is a Cayley graph of the cyclic group $\mathbb{Z}_n$.



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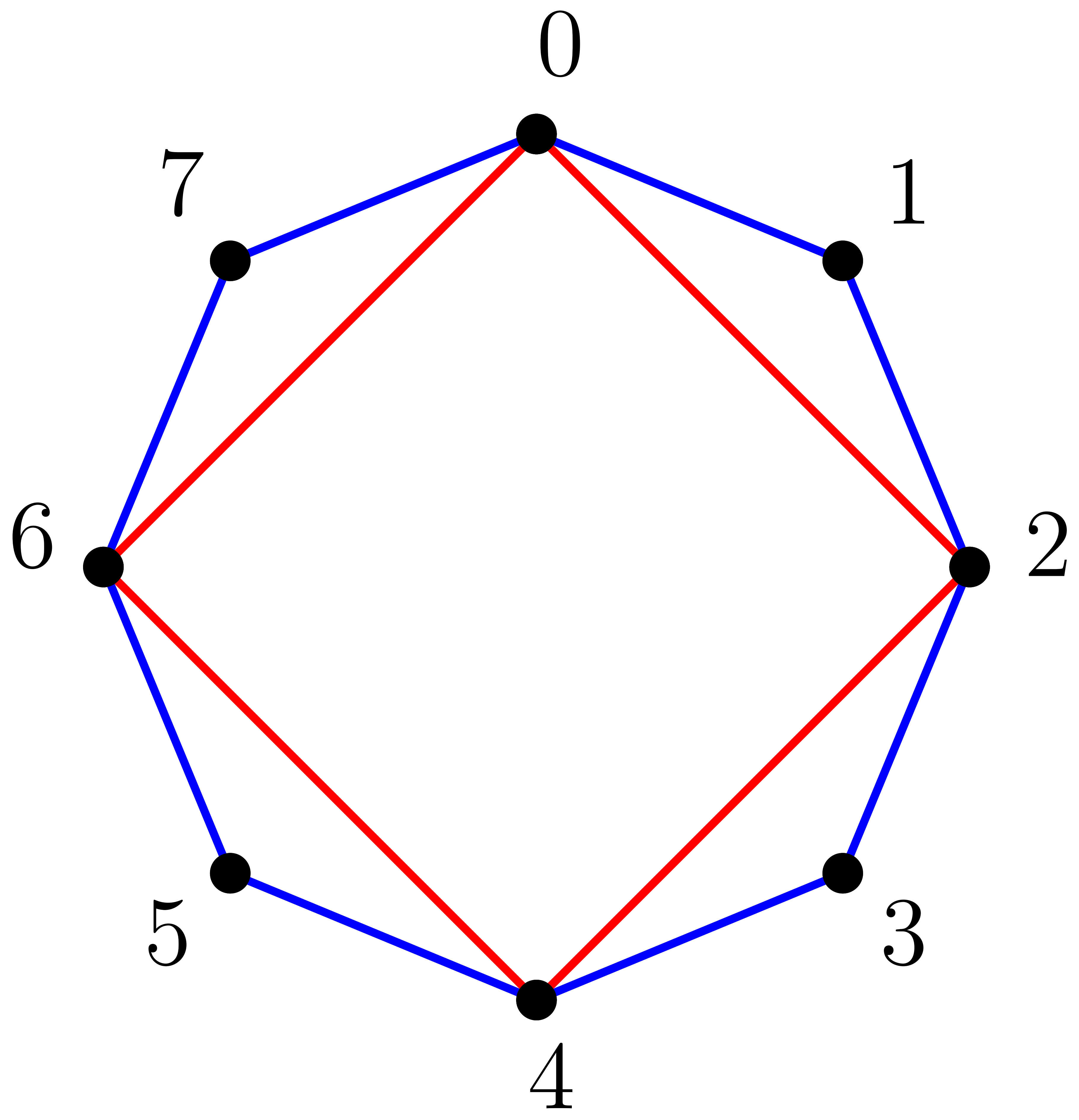

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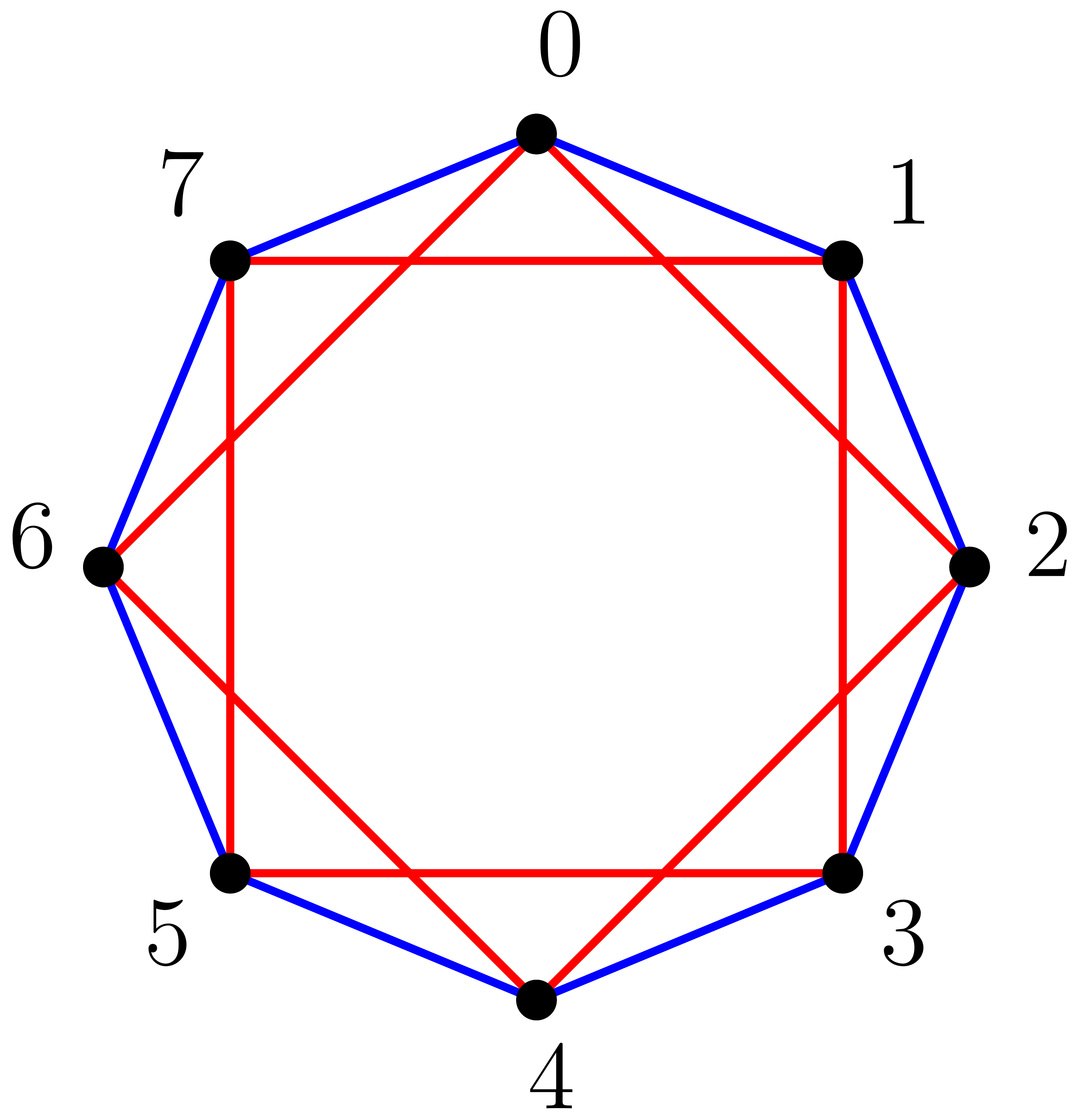

$\mathrm{Circ}(8, \{\pm 1, \boxed{\pm 2}\})$

Wilson conditions

=

sufficient conditions for a circulant graph to be unstable

Wilson condition (C.4)

$X = \mathrm{Circ}(n, S)$, n even

If there exists an $m \in \mathbb{Z}_n^\times$ such that $\frac{n}{2} + mS = S$, then X is unstable.

$$\phi(x, i) = \begin{cases} (mx, 0) & \text{if } i = 0 \\ \left(mx + \frac{n}{2}, 1\right) & \text{if } i = 1 \end{cases}$$

$$\phi \in \mathrm{Aut}(X \times K_2)$$

$$\phi \notin \mathrm{Aut}(X) \times \mathrm{Aut}(K_2)$$

Corrections of Wilson conditions

- **Qin-Xia-Zhou** (2019) updated Wilson condition (C.2) to (C.2$'$).
- **Hujdurović-Mitrović-Morris** (2021) updated Wilson condition (C.3) to (C.3$'$).

Wilson’s conjecture

Every **non-trivially unstable circulant graph** satisfies at least one of the **Wilson conditions**.

Circulants of odd order

Theorem (Fernandez-Hujdurović 2022)

There are no non-trivially unstable circulants of odd order.

Wilson's conjecture is vacuously true for circulants of odd order!

Circulants of order twice an odd prime

Theorem (Hujdurović-Mitrović-Morris 2021)

Every non-trivially unstable circulants of **order** $2p$, p an odd prime,

satisfies **Wilson condition** $(C.4)$.

Reminder (Wilson condition $(C.4)$)

$$\frac{n}{2} + mS = S \text{ with } m \in \mathbb{Z}_n^\times$$

Circulants of order twice an odd prime

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Every non-trivially unstable circulants of **order** $2p$, p an odd prime,

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Wilson's conjecture is true for

circulants of order twice a prime!

Circulants of low valency

Theorem (Hujdurović-Mitrović-Morris 2023+)

Every non-trivially unstable circulants of **valency at most 7** satisfies **at least one Wilson condition**.

Wilson's conjecture is true for circulants of valency at most 7!

Circulants of low valency

Theorem (Hujdurović-Mitrović-Morris 2023+)

Every non-trivially unstable circulants of **valency at most 7** satisfies **at least one Wilson condition**.

A **classification** has been obtained for **each valency at most 7**.

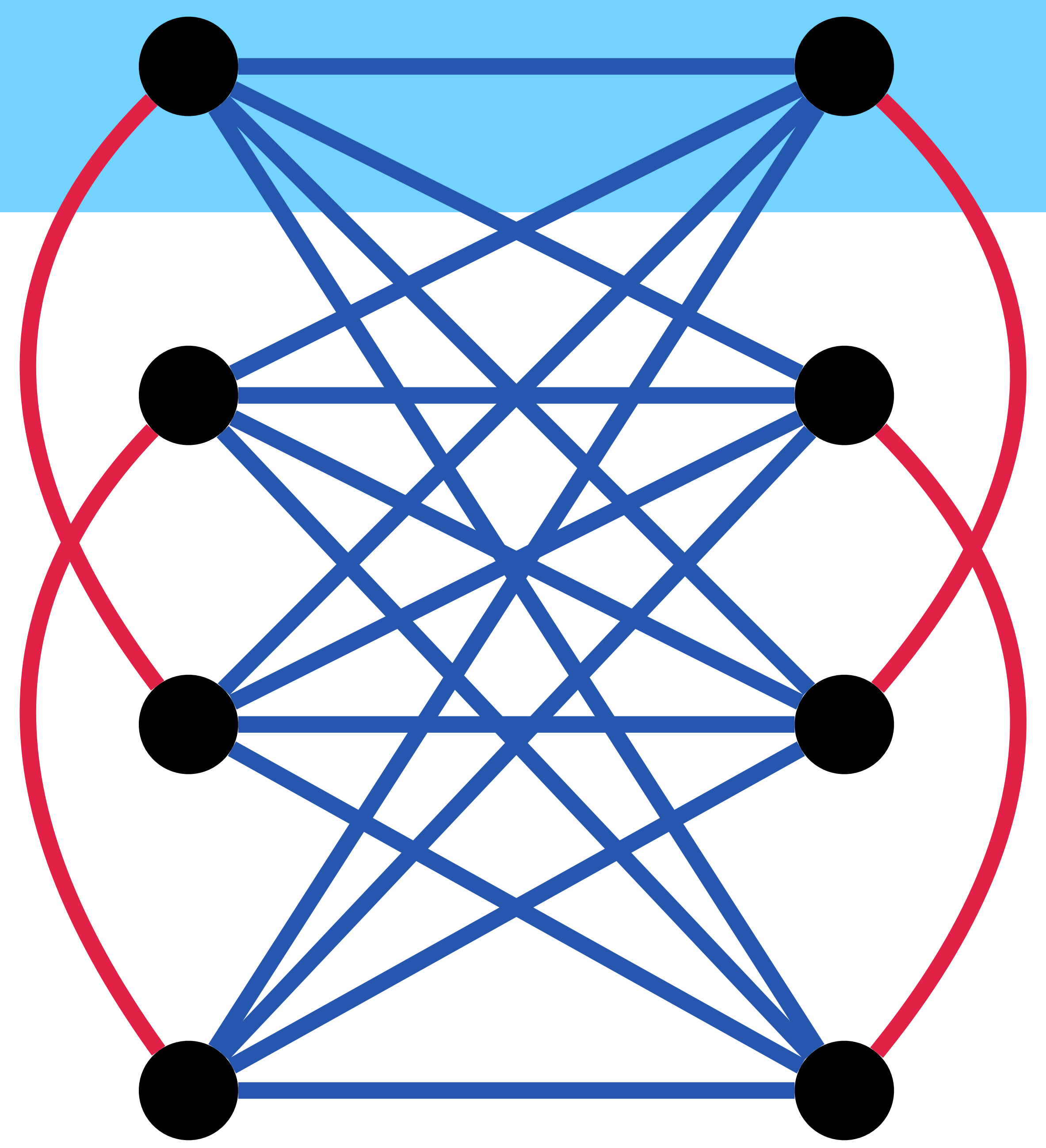
- For each valency, we provide a **complete list of connection sets**.
- For each graph, we find a **Wilson condition** it satisfies.

An example of a classification result

Theorem (Hujdurović-Mitrović-Morris 2023+)

A 5-valent circulant is unstable if and only if it is **trivially unstable** or one of the following

1. $\mathrm{Circ}(12k, \{\pm s, \pm 2k, 6k\})$ with s odd, satisfying **Wilson condition** $(C.1)$;
2. $\mathrm{Circ}(8, \{\pm 1, \pm 3, 4\})$ satisfying **Wilson condition** $(C.3')$.



Non-trivially unstable circulants of low valency

- **valency** ≤ 3: **none**
- **valency** 4: **two** infinite families satisfying $(C.4)$
- **valency** 5: **one** infinite family $(C.1)$; **one sporadic** example $(C.3')$
- **valency** 6: **seven** infinite families $(C.1) - (C.4)$
- **valency** 7: **six** infinite families $(C.1) - (C.3')$

Theorem (Hujdurović-Mitrović-Morris 2023+)

Every non-trivially unstable circulants of **valency at most 7** satisfies **at least one Wilson condition**.

This bound is sharp!

$\mathrm{Circ}(48, \{\pm, 3, \pm 4, \pm 6, \pm 21\})$

1. 8-valent
2. non-trivially unstable
3. **does not satisfy any of the Wilson conditions**

$\mathrm{Circ}(48, \{\pm, 3, \pm 4, \pm 6, \pm 21\})$

1. 8-valent
2. non-trivially unstable
3. **does not satisfy any of the Wilson conditions**

Wilson's conjecture is false in general!

Generalisations of Wilson conditions

Generalisation of the Wilson condition $(C.4)$

Theorem (Hujdurović-Mitrović-Morris 2021)

If $X = \mathbf{Circ}(n, S) \cong \mathbf{Circ}\left(n, S + \frac{n}{2}\right)$ then X is unstable.

Generalised Wilson condition $(C.4)$

Theorem (Hujdurović-Mitrović-Morris 2021)

If $X = \mathbf{Circ}(n, S) \cong \mathbf{Circ}\left(n, S + \frac{n}{2}\right)$ then X is unstable.

For $\ell \geq 4$, consider $X = \mathrm{Circ}(n, S)$ with

$$n = 3 \cdot 2^{\ell} \text{ and } S = \{\pm 3, \pm 6, \pm \frac{n}{12}, \frac{n}{2} \pm 3\}$$

1. X is **8-valent** and **non-trivially unstable**.
2. X satisfies the **Generalised Wilson condition** $(C.4)$.
3. **X does not satisfy any of the original Wilson conditions.**

Other generalisations

$X = \mathrm{Circ}(n, S)$

$H, K \leq \mathbb{Z}_n$ are non-trivial with $|K|$ even; $K_o = K \backslash 2K$.

Theorem (Hujdurović-Mitrović-Morris 2021)

If either

- $S + H \subseteq S \cup (K_o + H)$ and $H \cap K_o = \varnothing$, or
- $(S \backslash K_o) + H \subseteq S \cup K_o$, and either $|H| \neq 2$, or $|K|$ is divisible by 4,

then X is **unstable**.

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The above result generalises Wilson conditions $(C.1)$, $(C.2')$ and $(C.3')$.

Each of the Wilson conditions $(C.1)$, $(C.2')$, $(C.3')$, $(C.4)$ has been generalised.

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Theorem (Hujdurović-Mitrović-Morris 2021)

If $X = \mathbf{Circ}(n, S) \cong \mathbf{Circ}\left(n, S + \frac{n}{2}\right)$ then X is unstable.

Check the paper for more!

Computational results

Every non-trivially unstable circulant of order **at most** 50 satisfies at least one generalisation we introduced.

Recent developments

Analogues of the Wilson conjecture for other graph families turned out to be true!

- **Generalised Petersen graphs** - Qin, Xia and Zhou (2020)
- **Toroidal graphs and Triangular grids** - Dave Witte Morris (2023)
- **Rose-Window graphs** - Ahanjideh, Kovács, Kutnar (2023)

Background

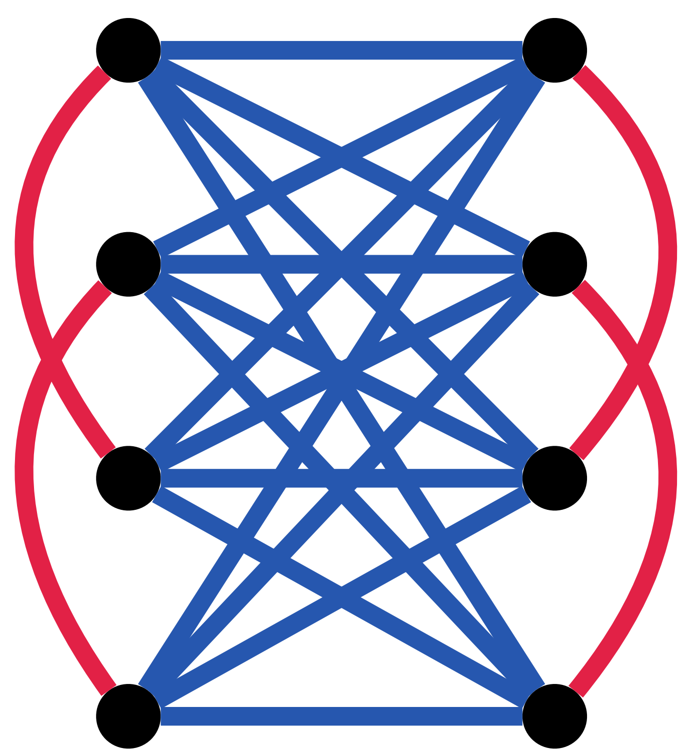
- $X \times K_2$ plays a major role in understanding $\mathrm{Aut}(X \times Y)$ for X non-bipartite and Y bipartite
- X is **non-trivially unstable** if it is connected, non-bipartite, twin-free, and $\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}(X) \times \mathrm{Aut}(K_2)$
- **Wilson's conjecture**: Every non-trivially unstable circulant graph satisfies at least one Wilson condition.

Results

- **Generalisations of Wilson conditions**
- New infinite families of **counterexamples** to Wilson's conjecture
- Wilson's conjecture is **true** for
    - **circulants of order $2p$**
    - **circulants of valency at most 7**

Questions?

Thank you for your attention!



Additional slides

Wilson conditions

$X = \mathrm{Circ}(n, S)$, n even. $S_e = S \cap 2\mathbb{Z}_n$, $S_o = S \backslash S_e$

1. There exists a non-zero element $h \in 2\mathbb{Z}_n$ such that $h + S_e = S_e$.
2. n is divisible by 4, and there exists $h \in 1 + 2\mathbb{Z}_n$ such
   - $2h + S_o = S_o$,
   - $\forall s \in S$ **with** $s \equiv 0$ **or** $-h \pmod 4$**, we have** $s + h \in S$**.**
3. There exists a subgroup $H \leq \mathbb{Z}_n$ such that the set

$$R = \{s \in S \mid s + H \nsubseteq S\},$$

is non-empty and has the property that if $d = \gcd(R \cup \{n\})$, then $\dfrac{n}{d}$ is even, $\dfrac{r}{d}$ is odd for all $r \in R$, and either $H \nsubseteq d\mathbb{Z}_n$ **or** $H \subseteq 2d\mathbb{Z}_n$.

4. There exists $m \in \mathbb{Z}_n^\times$, such that $\dfrac{n}{2} + mS = S$.