# Quadratic Forms in Design Theory

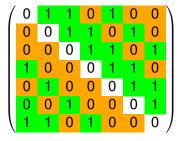
Padraig Ó Catháin

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- Joint work with Oliver Gnilke, Oktay Olmez & Guillermo Nunez Ponasso
- Inspired by a problem of Darryn Bryant
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## The symmetric mosaic problem

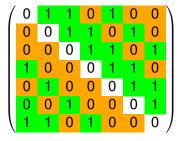


**Question:** For which parameters does there exist a mosaic of symmetric designs?

- A symmetric balanced incomplete-block design (SBIBD, design) with parameters (v, k, λ) has v points and v blocks. Each block is incident with k points, and each pair of points are jointly incident with λ blocks.
- Finite projective planes are designs with parameters  $(n^2 + n + 1, n + 1, 1)$ .
- A (v, k, λ) design is described by its incidence matrix, which is a square {0, 1}-matrix satisfying

$$MM^{\top} = (k - \lambda)I_{\nu} + \lambda J_{\nu}$$

## The symmetric mosaic problem



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$$(M + I)(M + I)^{\top} = MM^{\top} + M + M^{\top} + I = \alpha I + \beta J$$
  
so  $M + M^{\top} = J - I$  and *M* is *skew*.

If M and M + I are both symmetric designs, then M is the incidence matrix of a skew-Hadamard design.

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 Such designs exist when 4t – 1 is a prime power. Conjectured to exist for all integers 4t – 1.

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• Bruck-Ryser-Chowla (easy part): If v is even then  $k_i - \lambda_i$  is a square.

**Proof:** det $(MM^{\top})$  = det $((k - \lambda)I + \lambda J) = k^2(k - \lambda)^{\nu-1}$  is square.

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- **Conjecture:** There are no even symmetric mosaics (on three colours).
- Before the end of the talk, we'll rule out the displayed example.

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$$\oplus (n^2+n+1, \frac{n^2+n}{2}, \frac{n^2+n-2}{4}) \oplus (n^2+n+1, \frac{n^2-n}{2}, \frac{n^2-3n+2}{4})$$

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- Some other parameters up to 1,000. Smallest example we can't rule out is n = 5 above.
- **Question:** Can the complement of a projective plane of order 5 be partitioned into a (31, 15, 7) and a (31, 10, 3)-design? (Both are known to exist individually.)

#### Theorem

Suppose that M is the incidence matrix of a symmetric  $(v, k, \lambda)$  design where v is odd. Then the Diophantine equation

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has a non-trivial solution.

• Question: How do I solve such equations?

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- Question: How do I solve such equations?
- Marshall Hall: the computations involved are detailed and troublesome.
- Question: What does this have to do with design theory?
- Question: Given a symmetric positive definite matrix *G*, when does there exist a rational matrix *M* such that  $MM^{\top} = G$ ?

#### Definition

A *quadratic form* is a (multivariate) polynomial in which every term has degree 2.

$$5x^{2} + 14xy + 10y^{2} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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• Linear substitution of variables yields an equivalence operation on forms:  $x_0 = x + \frac{9}{5}y$  and  $y_0 = 2x + \frac{13}{5}y$  gives

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• Yields a rational matrix factorisation:

$$MM^{\top} = \begin{pmatrix} 1 & 2 \\ rac{9}{5} & rac{13}{5} \end{pmatrix} \begin{pmatrix} 1 & rac{9}{5} \\ 2 & rac{13}{5} \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}$$

$$x^{2} + 4xy + 6xz + 4y^{2} + 10yz - z^{2} \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & -1 \end{bmatrix}$$

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Polarisation is no harder than Gaussian elimination. Every quadratic form can be **polarised**.  $S \sim x_0^2 - 10y_0^2 - 10z_0^2 \sim \langle 1, -10, -10 \rangle$ 

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Quadratic forms are **congruent** if there exists an invertible linear substitution of variables from one form to the other. If matrices S and T represent the forms, then there exists invertible M such that

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- **Sylvester:** (Symmetric) Matrices over  $\mathbb{R}$  are congruent if and only they have the same number of positive and negative eigenvalues.

## Quadratic forms

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- **Sylvester:** (Symmetric) Matrices over  $\mathbb{R}$  are congruent if and only they have the same number of positive and negative eigenvalues.
- Over  $\mathbb{Q}$  the question is harder (because  $\mathbb{Q}^*/\mathbb{Q}^{*2}$  is infinite).

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- If we show nl + J is not a Gram matrix, certain projective planes will not exist.
- If S is a Gram matrix, det(S) is a square. Discriminant = 1
- If *S* is a Gram matrix its eigenvalues are positive. **Positive Definite**
- These conditions are not sufficient.

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• Polarise *S*, since it has discriminant 1, get  $\langle a_0, n^2 a_0 \rangle$ .

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So  $\langle a, a \rangle = \langle 1, 1 \rangle$  if and only if *a* is a sum of two squares.

• Fermat: An integer *a* is a sum of two squares if and only if no prime  $p \equiv 3 \mod 4$  divides the square free part of *a*.

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For prime *p* and integer *a*, a *Legendre symbol* is defined to be  $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p$ . It is 1 if *a* is a quadratic residue and -1 otherwise.

#### Definition

For prime *p* and integers *a*, *b*, a *Hilbert symbol* is defined to be  $(a, b)_p = 1$  if  $aX^2 + bY^2 = Z^2$  has a solution (in the *p*-adics). It is -1 otherwise.

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$$(a,p)_p = \left(\frac{a}{p}\right).$$

•  $(p,p)_p = \left(\frac{-1}{p}\right)$  this is 1 if  $p \equiv 1 \mod 4$  and -1 if  $p \equiv 3 \mod 4$ .

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- $(p,p)_p = \left(\frac{-1}{p}\right)$  this is 1 if  $p \equiv 1 \mod 4$  and -1 if  $p \equiv 3 \mod 4$ .
- $(ab, c)_p = (a, c)_p (b, c)_p$  the Hilbert symbol is bilinear.

#### Theorem

Suppose that Q is a quadratic form in two variables, which polarises to  $\langle a, a \rangle$ . Then Q is congruent to  $x^2 + y^2$  if and only if  $(a, a)_p = 1$  for every prime p.

#### Proof.

Suppose p divides the square-free part of a. Then

$$(a,a)_p = (-1,a)_p = \left(\frac{-1}{p}\right)$$

which is -1 if and only if  $p \equiv 3 \mod 4$  by **Gauss**. So  $\langle a, a \rangle = \langle 1, 1 \rangle$  if and only if no prime 3 mod 4 divides the square-free part of *a*. This is if-and-only-if *a* is a sum of two squares by **Fermat**.

#### Theorem (Two dimensional Hasse-Minkowski)

A symmetric matrix G is a Gram matrix if and only if

- It is positive definite.
- It has discriminant 1.
- For some (in fact, any) polarisation G = ⟨a, a⟩, all the Hilbert symbols (a, a)<sub>p</sub> are 1.

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- For some (in fact, any) polarisation G = ⟨a, a⟩, all the Hilbert symbols (a, a)<sub>p</sub> are 1.
- This is all computationally easy, and very concrete.
- The Hilbert symbol is bilinear, which simplifies the construction of invariants in higher dimensions.
- Gnilke, Ó C., Olmez, Ponasso: Invariants of Quadratic Forms and applications in Design Theory, LAA, 2024.

Let *Q* be a quadratic form, equivalent to the polarisation  $\langle a_1, a_2, ..., a_n \rangle$ . The *Hasse-Minkowski invariant* of *Q* at the prime *p* is

$$HM(Q,p) = \prod_{i < j} (a_i, a_j)_p$$
.

#### Theorem (Hasse-Minkowski, easy part)

A symmetric matrix G is a Gram matrix (if and) only if

- It is positive definite.
- It has discriminant 1.
- For some (in fact, any) polarisation G = ⟨a<sub>1</sub>, a<sub>2</sub>,..., a<sub>n</sub>⟩, the invariants HM(Q, p) are 1 for all (odd) primes p.

# Hasse-Minkowski is neither detailed nor troublesome (mostly)

 $\langle 5,7,21,15\rangle = (5,7)(5,21)(5,15)(7,21)(7,15)(21,15)$ = (5,7)(5,7)(5,3)(5,3)(5,5)(7,3)(7,7)...

= (3,3)(3,5)(3,7)(5,5)(5,7)

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For p = 5, this evaluates to

$$(3,5)_5(5,5)_5 = \left(\frac{3}{5}\right)\left(\frac{-1}{5}\right) = -1 \cdot 1$$

So  $\langle 5,7,21,15\rangle$  and  $\langle 1,1,1,1\rangle$  are **not** congruent.

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So (5, 7, 21, 15) and (1, 1, 1, 1) are **not** congruent. **Legendre:**  $(n, n, n, n) \sim (1, 1, 1, 1)$  for any integer *n*.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 25 & 0 & 0 & 0 & 0 \\ 0 & 2 \cdot 4 & 0 & 0 & 0 \\ 0 & 0 & 6 \cdot 4 & 0 & 0 \\ 0 & 0 & 0 & 12 \cdot 4 & 0 \\ 0 & 0 & 0 & 0 & 20 \cdot 4 \end{pmatrix} \sim \begin{pmatrix} 25 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

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- The pattern generalises, simplifies to  $\langle k^2, n, n, \ldots \rangle$  where  $n = k \lambda$ .
- Properties of Hilbert symbols simplify this to  $(n, n)^{\binom{v-1}{2}}$ .

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- Properties of Hilbert symbols simplify this to  $(n, n)^{\binom{v-1}{2}}$ .
- Non-trivial condition if  $n \equiv 1,2 \mod 4$ , requires  $n = x^2 + y^2$ .
- Fermat: There is no projective plane of order 6 or 14 or ...

### Summary of quadratic forms

- The Hilbert symbols are **computationally easy** to compute (assuming that the matrix entries are factorised).
- Can decide effectively whether symmetric matrices are congruent.
- In contrast, Diophantine equations are typically hard to solve. BRC state their theorem in a way which avoids mention of congruence.

### Summary of quadratic forms

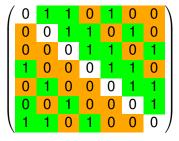
- The Hilbert symbols are **computationally easy** to compute (assuming that the matrix entries are factorised).
- Can decide effectively whether symmetric matrices are congruent.
- In contrast, Diophantine equations are typically hard to solve. BRC state their theorem in a way which avoids mention of congruence.
- Hasse-Minkowski theory is non-constructive: typically to not find any congruence matrix (let alone (0, 1)-congruence matrices).
- The hard (and non-constructive) part of the theorem shows that every global obstruction comes from a local obstruction.
- Deciding whether (k − λ)I + λJ = MM<sup>T</sup> has a rational solution is, in practice, easy. The condition is that

$$(k-\lambda,(-1)^{\nu-1/2}\lambda)_p=1$$

for all primes p.

Ó Catháin

## Back to the symmetric mosaic problem



**Question:** For which parameters does there exist a mosaic of symmetric designs?

#### Proposition

Suppose that  $M_1$ ,  $M_2$  and  $M_1 + M_2$  are incidence matrices of symmetric designs. Define  $Q = M_1 M_2^{\top} + I$ . Then  $QQ^{\top} = \sigma I + \tau J$  where  $\sigma = (k_1 - \lambda_1) (k_2 - \lambda_2) - \alpha + 1$  and  $\tau = v\lambda_1\lambda_2 + \lambda_2(k_1 - \lambda_1) + \lambda_1(k_2 - \lambda_2) + \alpha$ .

#### Theorem

If v is even then

$$(k_1 - \lambda_1)(k_2 - \lambda_2) - \frac{2k_1k_2}{v - 1} + 1$$

is the square of an integer. If v is odd, then

$$(\sigma,\sigma)_{\rho}^{\binom{\nu-1}{2}}(\sigma,\nu)_{\rho} = (\sigma,(-1)^{\nu-1/2}\nu)_{\rho} = 1$$

for all odd primes p.

Our theorem rules out the only even mosaic on less than 10,000 points

 $(2380, 183, 14) \oplus (2380, 793, 264) \oplus (2380, 1404, 828)$ .

because  $13^2 \times 23^2 - 11^2$  is not a square.

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• As in BRC, the result is weaker in the odd case, ruling out about half of possible parameter sets. It rules out decomposing the complement of a projective plane of order 9:

 $(91, 45, 22) \oplus (91, 36, 14) \oplus (91, 10, 1)$ .

The Hilbert symbol reduces to  $(471, 471)_p(471, 91)_p$ . At p = 3 this is  $(3,3)_p(3,1)_p = -1$ .

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But the theorem does not rule out existence of a

 $(31, 15, 7) \oplus (31, 10, 3) \oplus (31, 6, 1)$ .

## Go raibh maith agaibh!