New 2-closed groups that are not automorphism groups of digraphs

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Orbitals, 2-closure

- A permutation group G acting on a set Ω can act coordinate-wise on $\Omega \times \Omega$.
- Orbits of this action are known as *orbitals* of G.
- G is called 2-*closed* if there are no larger permutation groups on Ω with the same orbitals.
- The *rank* of G is the number of orbitals.

GRR and DRR problems

Which regular permutation groups are automorphism groups of (di)graphs?

- Active area of research during 1970s.
- GRR problem completed by Godsil[1] in 1978.
- DRR problem completed by Babai[2] in 1980.

[1] C. D. Godsil. GRRs for nonsolvable groups. In *Algebraic methods in graph theory, Vol. I, II (Szeged, 1978)*, Colloquia mathematica Societatis Janos Bolyai, 25, pages 221–239. 1981.

[2] L. Babai. Finite digraphs with given regular automorphism groups. *Periodica mathematica Hungarica*, 11(4):257–270, 1980.

Nonregular groups

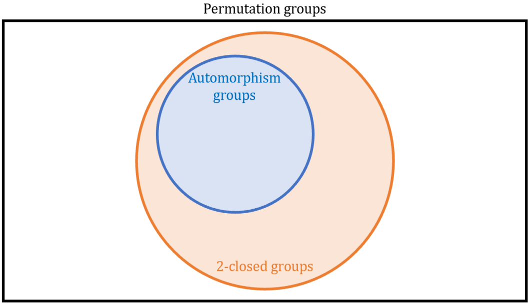
- Open problem: *Which integers are the degree of a 2-closed transitive group, but are not the order of any vertex-transitive non-Cayley graph?*
- "We should first find nonregular 2-closed groups that are not the full automorphism groups of (di)graphs. We do not know such examples." - Xu, 2008[3].

[3] M.-Y. Xu. A note on permutation groups and their regular subgroups. *Journal of the Australian Mathematical Society*, 85(2):283–287, 2008.

Nonregular groups

Which nonregular 2-closed permutation groups are not the automorphism group of any digraph?

- Giudici, Morgan and Zhou[4] recently solved this for primitive permutation groups of rank at most 4.



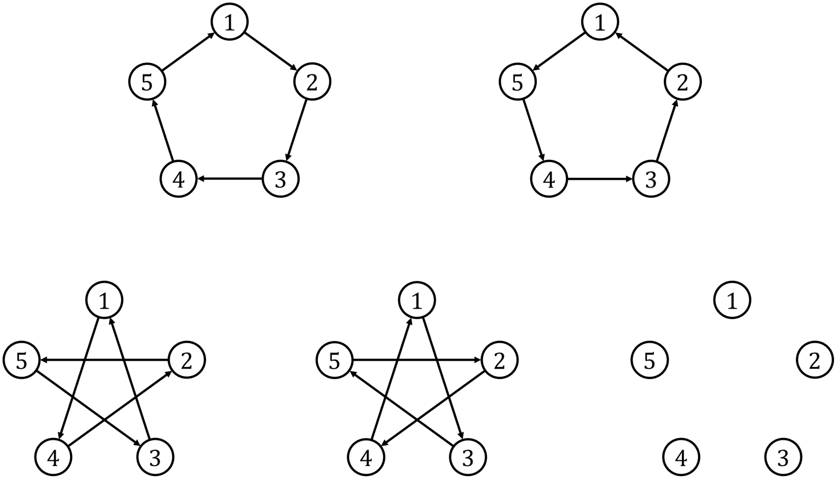
[4] M. Giudici, L. Morgan, and J.-X. Zhou. On primitive 2-closed permutation groups of rank at most four. *Journal of Combinatorial Theory. Series B*, 158:176–205, 2023.

Orbital digraphs

- If G is permutation group on Ω, then each orbital B corresponds to an *orbital digraph* with vertex set Ω and arc set B.
- If G is the automorphism group of a digraph Γ, then Γ is a union of orbital digraphs of G.

Orbital digraphs

Let $G = \langle (12345) \rangle \leqslant S_5$.



Nonregular 2-closed groups that are not automorphism groups of digraphs

- Let F be a finite field of odd order q.
- Let V and W be 2- and m-dimensional F-spaces, respectively.
- Let $D_8 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$ act on V.
- Let $G(m, q) = (V \otimes W) \rtimes (D_8 \circ \mathrm{GL}(W))$.

Orbital digraphs of $G(m, q)$

- Γ_B: Two vertices $x, y \in V \otimes W$ are adjacent precisely if their difference is not a simple tensor.
- Γ_λ (for each $\lambda \in F$): Two vertices $x, y \in V \otimes W$ are adjacent precisely if their difference lies in
$$\left\{(1, \pm\lambda) \otimes w, \left(1, \pm\lambda^{-1}\right) \otimes w \mid w \in W \setminus \{0\}\right\}.$$

Orbital digraphs of $G(m, q)$

- Γ_0 and Γ_1 are isomorphic to the Hamming graph $H(2, q^m)$.
- Γ_i is isomorphic to the Hamming graph $H(2, q^m)$ whenever $i^2 = -1$.
- This Hamming graph has automorphism group $S_{q^m} \wr S_2 \neq G(m, q)$.

Orbital digraphs of $G(m, q)$

- Every digraph has the same automorphism group as its complement.
- Each union of orbital digraphs that *does not* include Γ_0 is the complement of a union that *does* include Γ_0.

It therefore suffices to check only the orbital digraph unions that do not include Γ_0.

Example: $G(m, 13)$

Union	Automorphism	Union	Automorphism
Γ_1	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \circ I$	$\Gamma_2 \cup \Gamma_5$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \circ I$
Γ_2	$\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix} \circ I$	$\Gamma_3 \cup \Gamma_5$	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \circ I$
Γ_3	$\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix} \circ I$	$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \circ I$
Γ_5	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \circ I$	$\Gamma_1 \cup \Gamma_2 \cup \Gamma_5$	$\begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix} \circ I$
$\Gamma_1 \cup \Gamma_2$	$\begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix} \circ I$	$\Gamma_1 \cup \Gamma_3 \cup \Gamma_5$	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \circ I$
$\Gamma_1 \cup \Gamma_3$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \circ I$	$\Gamma_2 \cup \Gamma_3 \cup \Gamma_5$	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \circ I$
$\Gamma_1 \cup \Gamma_5$	$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \circ I$	$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \circ I$
$\Gamma_2 \cup \Gamma_3$	$\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix} \circ I$		

2-closure of $G(m, q)$

- A group is 2-closed if it is equal to the intersection of its orbital digraphs' automorphism groups.
- If $\lambda \in F$ such that $\lambda^2 \notin \{0, \pm 1\}$, then Γ_λ is the union of two Hamming graphs. If q is prime, it can be shown that $\mathrm{Aut}(\Gamma_\lambda) \leqslant (V \otimes W) \rtimes (\mathrm{GL}(V) \circ \mathrm{GL}(W))$.
- The intersection of the automorphism groups can then be found computationally.

Results

- If $m \geqslant 2$, then the groups $G(m, 5)$, $G(m, 7)$ and $G(m, 13)$ are 2-closed groups that are not the automorphism group of any digraph.
- These groups have rank 5, 5 and 7 respectively.
- These are the first known examples of nonregular 2-closed groups of rank greater than 4 that are not automorphism groups of digraphs.

Prime field orders

- If $G(m, p)$ is not the automorphism group of any digraph, then $p \in \{3, 5, 7, 13, 17\}$.
- However, $G(m, 17)$ is the automorphism group of $\Gamma_1 \cup \Gamma_2$.

Hence $G(m, p)$ is a 2-closed group that is not the automorphism group of any digraph if and only if $p \in \{3, 5, 7, 13\}$.