Combinatorial results for certain semigroups of contraction mappings of a finite chain

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- Abstract
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- The Symmetric Inverse Monoid/Semigroup
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- The algebraic results show that these semigroups are nonregular (left) abundant semigroups (for $n \geq 4$) whose Green's relations admit a nontrivial characterization.
- The combinatorial enumeration results show links with Fibonacci numbers, Motzkin numbers and sequences some of which are in the encyclopedia of integers sequences (OEIS).

Historical Background

- Recall that a *group* is a set G with an operation $*$, satisfying the following:

  1. $\forall x, y \in G,\ x * y \in G$;
  2. $\forall x, y, z \in G,\ (x * y) * z = x * (y * z)$;
  3. $\exists e \in G$ with $e * x = x * e$, $\forall x \in G$.
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- A *semigroup* is a set S with an operation *, satisfying (1) and (2) above. If it satisfies (3) also then it become a monoid.

Historical Background

There are many classes of semigroups (much more than groups and rings) the most notable of which are: von Neumann regular, inverse, orthodox, eventually regular, group bound, bands and semibands. Others are: abundant, adequate, ample, quasi adequate, nil, nilpotent, band, semibands and idempotent-free semigroups.

Historical Background

Some examples:

- $(Z, +)$ is a group / monoid / semigroup.
- $(Z, \times)$ is NOT a group but a monoid.
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Further Examples:

- $\mathcal{I}(X)$ is NOT a group but a monoid - THE SYMMETRIC INVERSE SEMIGROUP/MONOID.
- $B = \{(m, n) | m, n \in N\}$ with multiplication $(m, n) * (p, q) = (m - n + t, q - p + t)$, where $t = \max(n, p)$ is NOT a group but a monoid - THE BICYCLIC MONOID.

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The Symmetric Inverse Semigroup/Monoid

For a given (partial) mapping or transformation $\alpha : Y \subseteq X \to X\alpha \subseteq X$, we denote its domain by $\mathrm{Dom}\,\alpha$, its image set or range by $\mathrm{Im}\,\alpha$ and its set of fixed points by $F(\alpha)$. If $\mathrm{Dom}\,\alpha = X$ then α is called a *full* or *total* transformation, otherwise it is *strictly* partial.

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The set of all full transformations of X, denoted by $\mathcal{T}(X)$, is known as the *full symmetric semigroup*.

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The set of all partial one-to-one (more appropriately, two-to-two) mappings of X, denoted by $\mathcal{I}(X)$, is known as the *symmetric inverse semigroup*. Partial one-to-one maps are also known as *subpermutations.* [Cameron and Deza, 1978].

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This class of semigroups and its subsemigroups provide us with a rich source of 'natural' examples. However, they are worth studying in their right as naturally occurring (mathematical) objects. [Howie, 1987].

The Symmetric Inverse Semigroup/Monoid

Cayley's Theorem (1854/1870)

Every group G is isomorphic to a subgroup of the symmetric group acting on G.

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- with $a_0 = 1$.

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- Ideally, we would like to compute $F_{kmp} = F(n; k, m, p)$ for any S.

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Lemma

[20, Lemma 2.1] Let $X_n = \{1, 2, \ldots, n\}$ and $P = \{p, m, k\}$, where for a given $\alpha \in \mathcal{I}_n$, we set $p = h(\alpha), m = f(\alpha)$ and $k = w^+(\alpha)$. We also define $F(n; k) = F(n; p, k) = 1$ if $k = p = 0$. Then we have the following:

1. $n \geq k \geq p \geq m \geq 0$;
2. $k = 1 \Rightarrow p = 1$;
3. $p = 0 \Leftrightarrow k = 0$.

Combinatorial Results and Problems

Let $c(n; p)$ be the number of surjective partial derangements $\alpha : X_n \longrightarrow Y_p = \{1, 2, \ldots, p\}$. Then from [19] we see that

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Lemma

[19, Theorem 1] Let $c(n;p)$ and $F(n;k,m,p)$ be as defined above. Then

$$F(n;k,m,p) = \binom{k-1}{p-1} \binom{p}{m} c(n-m;p-m).$$

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A subpermutation $\alpha \in \mathcal{I}_n$ is said to be a *contraction* mapping if (for all $x, y \in \mathrm{Dom}\,\alpha$) $|x - y| \geq |x\alpha - y\alpha|$. The semigroup of all subpermutation contractions of X_n is denoted by $\mathcal{OCI}_n$.

A subpermutation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving* if (for all $x, y \in \mathrm{Dom}\,\alpha$) $x \leq y$ implies $x\alpha \leq y\alpha$. The semigroup of all order-preserving subpermutations of X_n is denoted by $\mathcal{OI}_n$.

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A subpermutation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving* if (for all $x, y \in \mathrm{Dom}\,\alpha$) $x \leq y$ implies $x\alpha \leq y\alpha$. The semigroup of all order-preserving subpermutations of X_n is denoted by $\mathcal{OI}_n$.

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Let $\mathcal{OCI}_n = \mathcal{OI}_n \cap \mathcal{CI}_n$; $\mathcal{ODCI}_n = \mathcal{OCI}_n \cap \mathcal{DI}_n$.

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Concluding Remarks

Further Subsemigroups

Let X be a POSET.

- Subsemigroups of order-preserving (reversing) transformations
- Subsemigroups of order-decreasing (increasing) transformations (Umar 1992, PhD thesis)
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Favourite Quote

Whenever you can, count.

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Vote of thanks

THANK YOU ALL!