Local actions and eigenspaces of vertex-transitive graphs

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Basic definitions

All graphs are finite, connected and simple (undirected, loopless, no multiple edges).

An arc in a graph is an ordered pair of adjacent vertices.

A graph Γ is G-vertex-transitive (G-arc-transitive) if $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the the vertex-set (arc-set) of Γ.

Arc-transitivity implies edge-transitivity and vertex-transitivity (under mild hypothesis).

Tutte's Theorem

Theorem (Tutte 1947)

If Γ is a 3-valent arc-transitive graph and v is a vertex of Γ, then $|\mathrm{Aut}(\Gamma)_v| \leq 48$.

Example

If Γ is the Petersen graph, then $|\mathrm{Aut}(\Gamma)_v| = 12$.

By Orbit-Stabiliser, $|\mathrm{Aut}(\Gamma)|$ grows linearly with respect to $|\mathrm{V}\Gamma|$:

$$|\mathrm{Aut}(\Gamma)| \leq 48|\mathrm{V}\Gamma|$$

Consequences

Tutte's Theorem allows one (e.g. Marston Conder) to construct 3-valent arc-transitive graphs of "small" order (up to 10000, say).

Theorem (Potočnik, Spiga, V 2017)

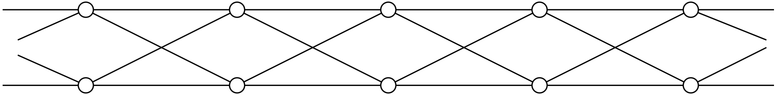
The number of 3-valent arc-transitive graphs of order at most n is

$$\sim n^{c \log n}$$

Many other results: structure, etc.

Failed generalisations

Let $\Gamma_r = \mathrm{C}_r[\overline{K}_2]$:



Γ_r is 4-valent and arc-transitive. Moreover $|\mathrm{Aut}(\Gamma_r)_v| \geq 2^r$ which is exponential in $|\mathrm{V}\Gamma_r| = 2r$.

There are related 3-valent vertex-transitive examples with $|\mathrm{Aut}(\Gamma)_v|$ exponential in $|\mathrm{V}\Gamma|$.

Local action

Let Γ be a G-vertex-transitive graph and let v be a vertex of Γ.

Let $G_v^{\Gamma(v)}$ denote the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$.

(Γ, G) is locally-L if $G_v^{\Gamma(v)}$ is permutation isomorphic to L.

Example

$(\mathrm{C}_r[\overline{K}_2], \mathrm{Aut}(\mathrm{C}_r[\overline{K}_2]))$ is locally-D_4. Also locally-$(\mathrm{Sym}(2) \wr \mathrm{Sym}(2))$.

Remark

Γ is G-arc-transitive if and only if $G_v^{\Gamma(v)}$ is transitive.

Graph-restrictive

A permutation group L is graph-restrictive if there exists a constant C such that if (Γ, G) is a locally-L pair, then $|G_v| \leq C$.

Tutte's Theorem: Transitive groups of degree 3 are graph-restrictive.

Gardiner (1973): All transitive groups of degree 4 except D_4 are graph-restrictive.

Weiss, Trofimov: 2-transitive groups and transitive groups of prime degree are graph-restrictive.

Conjecture (Potočnik, Spiga, V 2011)

Graph-restrictive $\iff$ *semiprimitive*

(A permutation group is semiprimitive if every normal subgroup is either transitive or semiregular.)

More general growth

We can (informally) define the "growth rate" of a permutation group L: in the class of locally-L pairs (Γ, G), how fast can $|G_v|$ grow as a function of $|\mathrm{V}\Gamma|$? (Least upper bound.)

Graph-restrictive $\iff$ "constant growth".

The fastest possible growth is exponential (for example D_4).

Problem

Find the growth rate of every permutation group.

A few results

Proposition (Potočnik, Spiga, V 2014)

(Non-trivial, imprimitive) Wreath products have exponential growth.

Theorem (Potočnik, Spiga, V 2014)

If a permutation groups has an imprimitivity system with two blocks and is not regular, then it has at least polynomial growth. Moreover, if the pointwise stabiliser of one block is not trivial, then it has exponential growth.

D_n has constant growth when n is odd, exponential growth when $n = 4$, and polynomial growth otherwise.

Small degree

There are 87 transitive groups of degree at most 8.

Theorem (as of 2019)

- 40 *have constant growth.*
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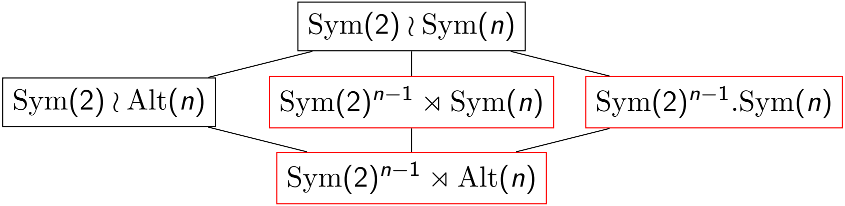
The groups of unknown growth



$\mathrm{Sym}(2)^n \rtimes \mathrm{Sym}(n) \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(n) \leq \mathrm{Sym}(2n)$.
This is the group preserving a partition of n parts of size 2.
(For example, the automorphism group of a set of n disjoint edges.)

$\mathrm{Sym}(2)^{n-1} \rtimes \mathrm{Sym}(n)$ is the same, except one is only allowed to flip an even number of edges.

The groups of unknown growth are the groups in red above, for $n = 3$ and $n = 4$.

Wreath groups have exponential growth

To show: $\mathrm{Sym}(2) \wr \mathrm{Sym}(n)$ has exponential growth.

Let (Γ_r, G_r) be an infinite family of locally-$\mathrm{Sym}(n)$ pairs.

Let $\Lambda_r = \Gamma_r[\overline{K}_2]$ and
$H_r = \mathrm{Sym}(2) \wr G_r = \mathrm{Sym}(2)^{|\mathrm{V}\Gamma_r|} \rtimes G_r \leq \mathrm{Aut}(\Lambda_r)$.

(Λ_r, H_r) is locally-$(\mathrm{Sym}(2) \wr \mathrm{Sym}(n))$ and $|H_r| = 2^{|\mathrm{V}\Gamma_r|}|G_r|$ is exponential in $|\mathrm{V}\Lambda_r|$.

Can handle any wreath product with minor changes.

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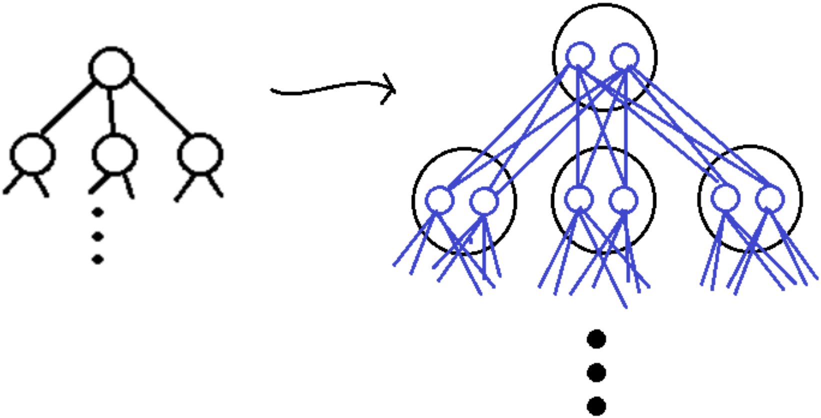
Now, $H_r = M \rtimes G_r \leq \mathrm{Sym}(2)^{|\mathrm{V}\Gamma_r|} \rtimes G_r$, for some M.

We need M a "large" subspace of $\mathrm{Sym}(2)^{|\mathrm{V}\Gamma_r|}$ preserved by G_r.

How to get correct local action?

Zero-sum around zero

How to make sure we get correct local action?



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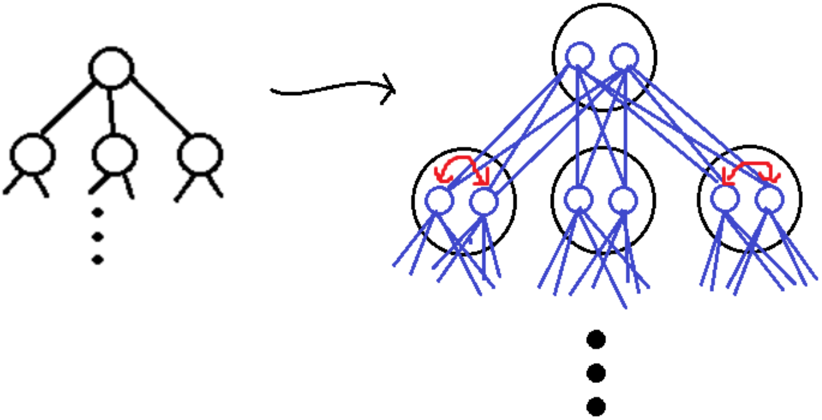

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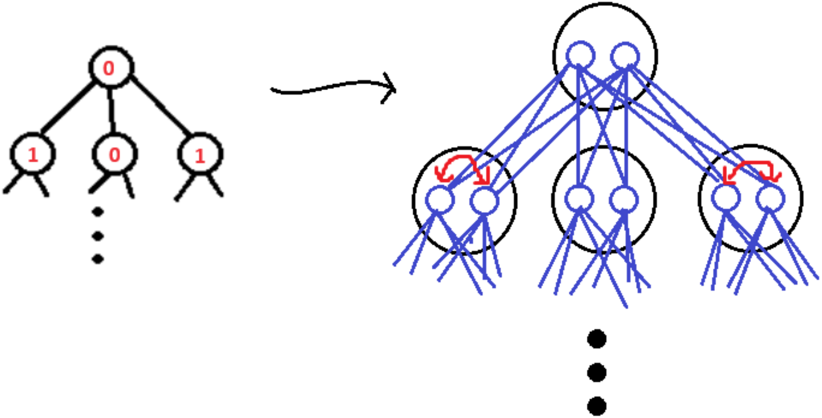

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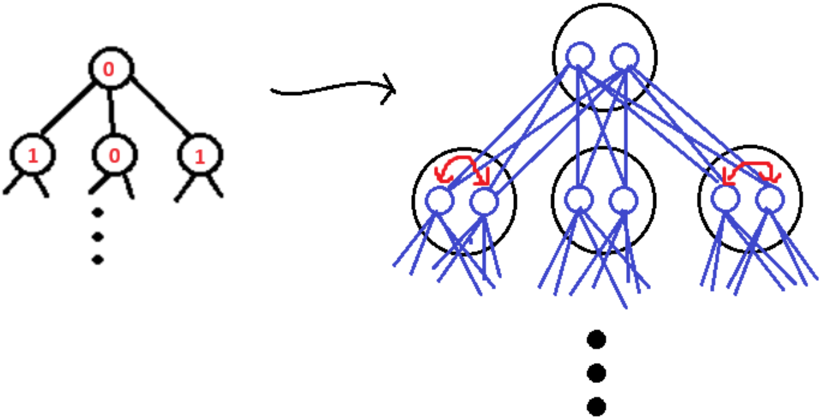

Conclusion: every element of M should have the "zero-sum around zero" property.

Zero-sum around zero

Let Γ be a graph and let

$$X_\Gamma = \left\{ x \in \mathbb{F}_2^{\mathrm{V}\Gamma} \colon\ (x(v) = 0) \Rightarrow \sum_{u \sim v} x(u) = 0 \right\}.$$

Note that X is not a subgroup/subspace, but we are looking for "large" groups contained in X (and preserved by a "nice" group).

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Eigenspaces (for the adjaceny matrix A_Γ) over $\mathbb{F}_2$.

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Proposition

If G is a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$, then the eigenspaces of A_Γ are exactly the maximal G-modules of X_Γ.

(Transitivity is important.)

Reformulation using eigenspaces

To show that $\mathrm{Sym}(2)^{n-1} \rtimes \mathrm{Sym}(n)$ has exponential growth, it suffices to find:

- an infinite family (Γ_r, G_r) of locally-$\mathrm{Sym}(n)$ pairs such that
- the dimension of some eigenspace over $\mathbb{F}_2$ of (the adjacency matrix of) Γ_r grows linearly with $|\mathrm{V}\Gamma_r|$.

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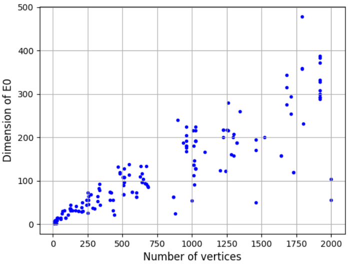
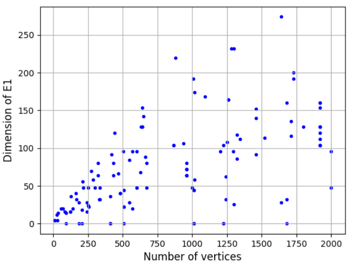
Let's get calculating!

Some data

Eigenspaces over $\mathbb{F}_2$ of locally-$\mathrm{Alt}(4)$ and $\mathrm{Sym}(4)$ graphs of order at most 2000 (Potočnik 2008):





Results

Theorem (Hujdurović, Potočnik, V (2021))

There exists locally-$\mathrm{Sym}(3)$ *3-valent graphs such that the dimension of the* 1*-eigenspace over* $\mathbb{F}_2$ *of the adjacency matrix grows linearly with the order of the graph.*

Corollary

$\mathrm{Sym}(2)^2 \rtimes \mathrm{Sym}(3)$ *has exponential growth.*

We also get the other two open groups of degree 6 with a small modification of the method.

Theorem (Mitrović, V (2023?))

There exists locally-$\mathrm{Sym}(4)$ *4-valent graphs such that the dimension of the* 0*-eigenspace over* $\mathbb{F}_2$ *of the adjacency matrix grows linearly with the order of the graph.*

Corollary

$\mathrm{Sym}(2)^3 \rtimes \mathrm{Sym}(4)$ *has exponential growth.*

The other two groups of degree 8 as well.

Constructions

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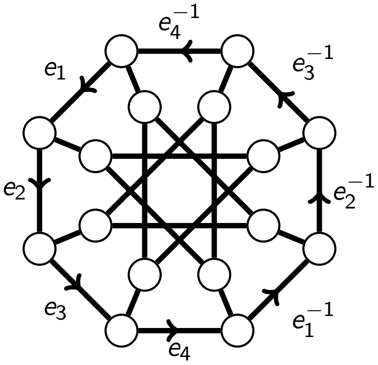
For example, in the 3-valent case, we take $\mathbb{Z}_n^4$-covers of the Möbius–Kantor graph.



The resulting graph has order $16n^4$. Computational evidence suggests that the 1-eigenspace over $\mathbb{F}_2$ has dimension $2n^4 + 2$ if n is odd and $2n^4 + 8$ if n is even.

Still much to learn

We are not even able to compute the dimensions of the eigenspaces for general n.

We use the “trick” of finding a small support eigenvector to get a (loose but still linear) lower bound on the dimension.

This is very ad hoc and relies on the voltages being very nice.

Still much to learn, II

We would like to generalise to other valencies, other "top" groups and other "bottom" (abelian) groups.

Random example:
$L = (C_3^2)^7 \rtimes \mathrm{PGL}(2,7) \leq C_3^2 \wr \mathrm{PGL}(2,7) \leq \mathrm{Sym}(9 \times 8)$.

Ideally, we would do this by learning how to "predict" the size of the eigenspaces of voltage covers...?

(At the moment, we don't even know which eigenspace we should be looking at.)

The dimensions often have nice (conjectured) forms.

For example, in the 4-valent case, we are taking $\mathbb{Z}_n^5$ voltage covers of a graph of order 30 (so the covers have order $30n^5$). The 0-eigenspaces over $\mathbb{F}_2$ seem to have dimension $6n^5 + 8$ when n is odd and $6n^5 + 32$ when n is even. It would be nice to be able to explain this.

Bigger picture: questions about growth

Are there groups of intermediate growth?

Problem

(Conjecturally) Characterize groups of polynomial growth.

SODO 2024

Symmetries of Discrete Objects, Auckland, February 12–16, 2024

- Symmetries of graphs, maps and polytopes
- Group action on trees
- Expander Cayley graphs

Organisers: Florian Lehner, Jeroen Schillewaert, Gabriel Verret

https://jschillewaert.wixsite.com/sodo2024