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Polytopes with minimal number of edges

Guillermo Pineda-Villavicencio,
Jie Wang, David Yost*

What I'm doing now

For a fixed dimension d, can we characterise all pairs

$$(v, e)$$

for which there is a d-polytope with v vertices and e edges?

$d = 3$ (Steinitz; 1906)

Theorem

There is a 3-polytope with v vertices and e edges if and only if

$$\frac{3}{2}v \leqslant e \leqslant 3v - 6.$$

$d = 4$ (Grünbaum; 1967)

Theorem

There is a 4-polytope with v vertices and e edges if and only if

$$2v \leqslant e \leqslant \binom{v}{2}$$

and $(v, e) \neq (6, 12), (7, 14), (10, 20)$ *or* $(8, 17)$.

$d = 4$ (Grünbaum; 1967)

Theorem

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A polytope is **simple** if the degree of every vertex is equal to the dimension, i.e. if

$$dv = 2e.$$

Why are there no simple 4-polytopes with 6, 7 or 10 vertices?

Denote by $\Delta_{m,n}$ the (direct) sum of an m-dimensional simplex Δ_m and an n-dimensional simplex Δ_n.

If P is a simple d-polytope with v vertices, then either

$v = d + 1$; P is a simplex Δ_d

$v = 2d$; P is a prism $\Delta_{1,d-1}$

$v = 3d - 3$; P is $\Delta_{2,d-2}$

$v = 3d - 1$; P is J_d (cut one vertex from a prism) or 3-cube

$v = 4d - 8$; P is $\Delta_{3,d-3}$

or $v >$ all the preceding numbers.

Simple polytopes with more than $4d-8$ vertices:

$v=4d-7$; $\Delta_{4,4}$, J_6

$v=4d-6$; $\Delta_{4,5}$, J_5

$v=4d-5$; $\Delta_{4,6}$, J_4

$v=4d-4$; $\Delta_{4,7}$, $\Delta_{5,5}$, $\Delta_{1,1,d-2}$, cut $\Delta_{2,d-2}$

$d = 5$ (independently by Kusunoki/Murai and Pineda-Villavicencio/Ugon/Yost)

Theorem

There is a 5-polytope with v vertices and e edges if and only if either
$2e = 5v$ *(i.e. simple) and* $v \neq 8$,
or

$$\frac{1}{2}(5v + 3) \leqslant e \leqslant \binom{v}{2}$$

and $(v, e) \neq (9, 25), (13, 35)$.

In particular, e cannot equal $\frac{1}{2}(5v + 1)$ or $\frac{1}{2}(5v + 2)$.
Why?

Define the **excess degree of a vertex** as its degree minus the dimension, and the **excess degree of a polytope** as the sum of the excess degrees of its vertices. The excess degree of a polytope is also equal to $2e - dv$.

Theorem

(Pineda-Villavicencio/Ugon/Yost) The excess degree of a nonsimple d-polytope is at least $d - 2$. Equivalently, there are no d-polytopes with excess degree in the range $[1, d - 3]$.

Idea of the proof: Suppose two facets F_1 and F_2 of a d-polytope P intersect in a face K which is not a ridge. If k is the dimension of K, then every vertex in K has at least $d-1-k$ neighbors in $F_1 \setminus K$, at least $d-1-k$ neighbors in $F_2 \setminus K$ and at least k neighbors in K. Thus each such vertex has excess degree at least $d-k-2$. Since K has at least $k+1$ vertices, the excess degree of P is at least $(k+1)(d-k-2)$, and this expression is at least $d-2$ for k in the range $[0, d-3]$. This *almost* establishes the Excess Theorem; the case when every two facets are either disjoint or intersect in a ridge requires separate treatment.

The excess degree is an important and natural concept, which encapsulates a lot of information about polytopes.

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It tells us that many values of (v, e) are not attained by any polytope.
And much more.

Theorem

(Pineda-Villavicencio/Ugon/Yost) If a d-polytope has $d + k$ vertices, and $k \leqslant d$, then its excess degree is at least

$$(k-1)(d-k),$$

and there is only one polytope with this minimal excess degree. If in addition $k \geqslant 4$, then every other such polytope has excess degree at least

$$(k-1)(d-k)+4.$$

In particular, there is no 4-polytope with 8 vertices and 17 edges, and there is no 5-polytope with 9 vertices and 25 edges.

The next result says that nonsimple vertices are gregarious.

Theorem

If a nonsimple vertex in a d-polytope has excess degree k, then it has at least $d - k - 2$ nonsimple neighbors.

This gives the alternative proof of the "Excess Theorem".

The minimum nonsimple value $d - 2$ implies a special structure.

Theorem

(Pineda-Villavicencio/Ugon/Yost) Any d-polytope P with excess exactly $d - 2$ either

1. *has a unique nonsimple vertex, which is the intersection of two facets, or*
2. *has $d - 2$ vertices of excess degree one, which form a $(d - 3)$-simplex which is the intersection of two facets.*

In particular, all nonsimple vertices have the same degree.

Polytopes with excess $d-2$ and not many vertices can be completely characterised.

Theorem

If P is a d-polytope with v vertices and excess $d-2$, then either

1. $v = d+2$; *P is a $(d-2)$-fold pyramid over a square;*
2. $v = 2d-1$; *P is pyramid over a prism* $\Delta_{1,d-2}$
3. $v = 2d+1$; *P is a pentasm*
4. $v = 3d-2$; *P is* C_d, Σ_d, N_d *or* A_4
5. $v = 3d-1$ *and* $d = 4$ *(three examples)*
6. *or* $v \geqslant 3d$.

For **excess** $d-1$, there is almost no restriction on the number of vertices, but severe restrictions on the dimension.

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Theorem

Excess $d-1$ *is possible only when* $d=3$ *or 5. Again, all nonsimple vertices have the same degree. When* $d=5$*, the nonsimple vertices form a face; either a single vertex, an edge, or a quadrilateral 2-face.*

In both dimensions, v can be any even number from $d+3$ onwards.

In contrast, **excess** d places severe restrictions on the number of vertices, but no restriction on the dimension.

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Theorem

Let P be a d-polytope with excess d.

1. *If $d \geqslant 7$, P is obtained by concatenating two simple polytopes along a simplex facet. In particular, P has d vertices with excess degree 1.*
2. *If $d = 5$, P is obtained by concatenating two simple polytopes along either a simplex facet or a pentagonal 2-face. Again, P has d vertices with excess degree 1.*
3. *If $d = 6$ and all nonsimple vertices have the same degree and they form a face.*
4. *If $d \neq 4$ or 6, then either $v = d + 2$, $v = 2d + 1$ or $v \geqslant 3d$.*

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3. *If $d = 6$ and all nonsimple vertices have the same degree and they form a face.*
4. *If $d \neq 4$ or 6, then either $v = d + 2$, $v = 2d + 1$ or $v \geqslant 3d$.*

This gives a new proof of the nonexistence of a 5-polytope with 13 vertices and 35 edges.

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Theorem

Excess $d+1$ is possible only when $d = 3$, 5 or 7. If $d = 3$ or 5, v can be any even number from $d+3$ onwards. If $d = 7$, the value $v = 14$ is excluded.

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Theorem

Suppose a d-polytope has excess $d+2$ where $d \geqslant 9$. Then $v = d+2$.

For the next range of values, the classification could not be simpler.

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Theorem

No d-polytope whatsoever has excess degree in the range $[d+3, 2d-7]$.

Finally, examples with excess $2d - 6$ are numerous.

Theorem

Fix d; then in any d-polytope with excess $2d - 6$, all nonsimple vertices have the same degree, either 1, 2, $d - 3$ or $2d - 6$. Such examples exist for all sufficiently large values of v, and also for the small values
$d + 3, 2d - 2, 2d + 2, 3d - 5, 3d - 3, 3d - 1, 3d + 1, 4d - 6, 4d.$

We conjecture that excess $2d-5$ is only possible if $d=3,5$ or 7. For higher values, the situations appears chaotic. There is a d-polytope with excess $2d-4$ in which the subgraph of nonsimple vertices is not even connected, let alone a face.

If we fix f_0, the question of minimising ξ is the same as minimising f_1.
For $d+3 \leq f_0 \leq 2d-2$, the minimum value of the excess degree is at least $2d-6 > d-2$.

$$\min\{\xi(P) : f_0 = d+k, k \leq d\} = (k-1)(d-k).$$

For $2d+1$ vertices, we have

$$\min\{\xi(P) : f_0 = 2d+1\} = d-2$$

except when $d=4$.
For $2d+2$ vertices, we have

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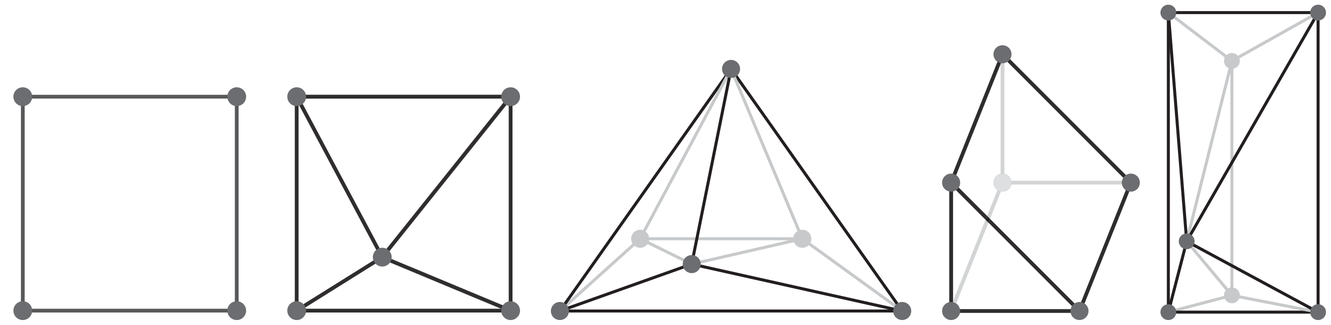
except in some low dimensions.
Problem: what about $2d+3$ vertices? $2d+4\ldots$?

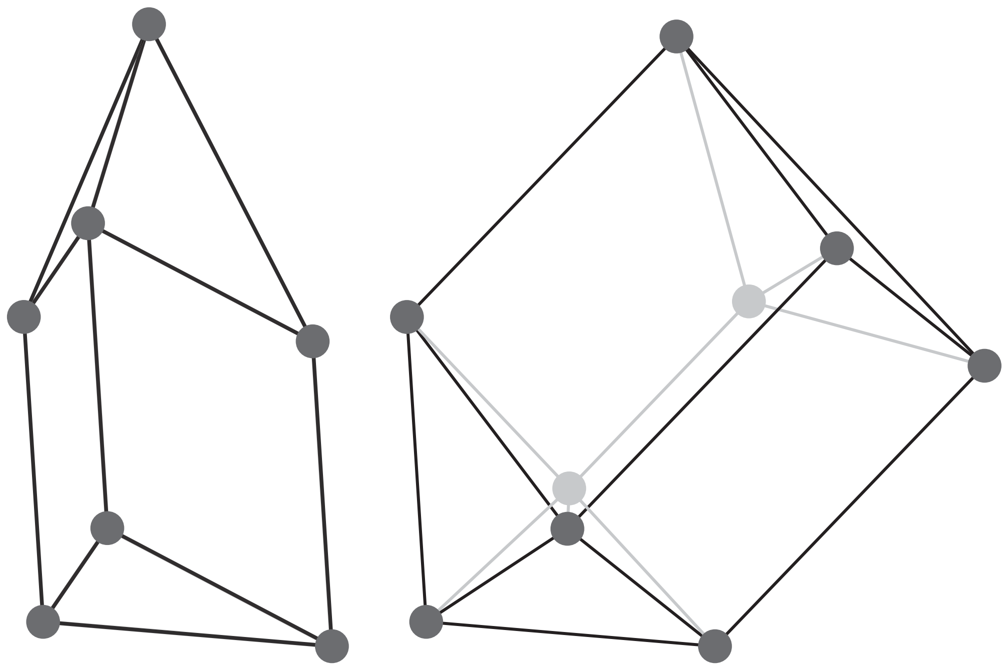
Theorem

Let P be a d-dimensional polytope with $d + k$ vertices, where $0 < k \leq d$.
(i) If P is a $(d - k)$-fold pyramid over the k-dimensional prism based on a simplex, then P has $\phi(d + k, d)$ edges.
(ii) Otherwise P has $> \phi(d + k, d)$ edges.
(iii) Furthermore, P has at least $d - k$ nonsimple vertices, with equality only if P is a $M(k, d - k)$-triplex



(a) $P_2 = M(2,0)$ (b) $M(2,1)$ (c) $M(2,2)$ (d) P_3 (e) $M(3,1)$

FIGURE 1. Triplices



(a) *Pentasm*3 (b) *Pentasm*4

FIGURE 2. Pentasms

Theorem

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(i) If P is a $(d-k)$-fold pyramid over the k-dimensional prism based on a simplex, then P has $\phi(d+k,d)$ edges.
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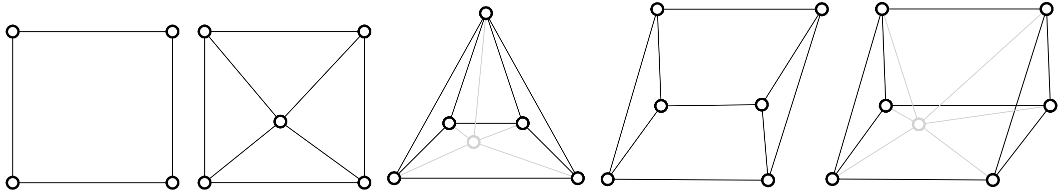
The polytope described in (i) will be denoted $M(k,d-k)$.
The proof depends on the identity

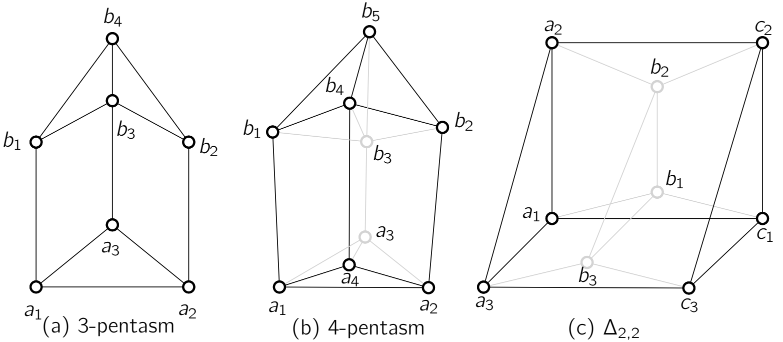
$$\phi(d+k-n,d-1)+nd-\binom{n}{2}=\phi(d+k,d)+(k-n)(n-2).$$

Note that if there are n vertices of a polytope lying outside a given facet, they must belong to at least $nd-\binom{n}{2}$ edges, and the facet must by induction contain sat least $\phi(d+k-n,d-1)$ edges.
Observe also that if P had strictly more than $2k$ simple vertices, then it would have strictly more than $\phi(d+k,d)$ edges.



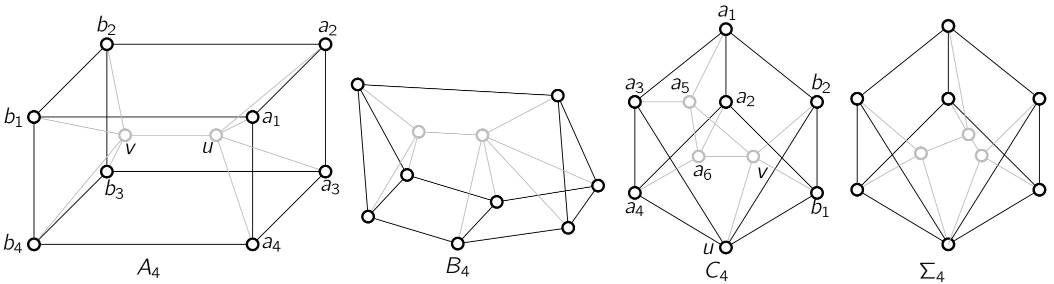
(a) $M_{2,0}$ (b) $M_{2,1}$ (c) $M_{2,2}$ (d) $M_{3,0}$ (e) $M_{3,1}$

Minimizers of the number of edges, for polytopes with no more than *2d* vertices



Minimizers of the number of edges, for polytopes with *2d+1* vertices

Polytopes with *2d+2* vertices with minimal number of edges



The 4-polytopes with ten vertices and 21 edges.

- B. Grünbaum, *Convex Polytopes*, Second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003,
- T. Kusunoki & S. Murai, *The Numbers of Edges of 5-Polytopes*, Ann. Comb. **23** (2019), no. 1, 89–101
- G. Pineda-Villavicencio, J. Ugon & D. Yost, *The Excess degree of a polytope*, SIAM J. Discrete Math. **32** (2018), no. 3, 2011–2046,
- G. Pineda-Villavicencio, J. Ugon & D. Yost, *Lower bound theorems for general polytopes*, European J. Combin. **79** (2019), 27–45,