Card Shuffle Groups

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Joint work with Binzhou Xia, Junyang Zhang and Wenying Zhu.

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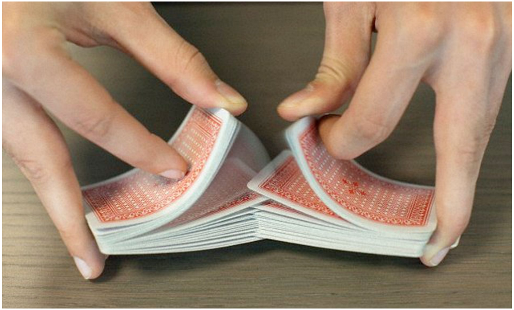
Shuffle Cards

We are motivated by an interesting paper[1] about mathematics in shuffling cards.



[1] P. Diaconis, R. L. Graham and W. M. Kantor, The mathematics of perfect shuffles., *Adv. Appl. Math.*, 4 (1983), 175–196.

Perfect Shuffle

- Cut the deck in half:

$$\begin{array}{r} 0 \text{---} \\ 1 \text{---} \\ \vdots \\ 2n-1 \text{---} \end{array} \longrightarrow \begin{array}{rl} 0 \text{---} & \text{---} n \\ 1 \text{---} & \text{---} n+1 \\ \vdots & \vdots \\ n-1 \text{---} & \text{---} 2n-1 \end{array}$$

- perfectly interleave them:

$$\begin{array}{rl} 0 \text{---} & \\ & \text{---} n \\ 1 \text{---} & \\ & \text{---} n+1 \\ \vdots & \\ n-1 \text{---} & \\ & \text{---} 2n-1 \end{array} \qquad \begin{array}{rl} n \text{---} & \\ & \text{---} 0 \\ n+1 \text{---} & \\ & \text{---} 1 \\ \vdots & \\ 2n-1 \text{---} & \\ & \text{---} n-1 \end{array}$$

Out-shuffle O In-shuffle I

Questions

Perform out-shuffles on a deck of 52 cards repeatedly.

Question: Can it return to the original order?

Answer: Yes. For example, after 52! times, since $O \in \mathrm{Sym}(52)$

Question: What is the minimum number of times needed to return to the original order?

Answer: 8 times.

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More patterns

- The order of the $2n$ cards after O is

$$(0, n, 1, n+1, \ldots, n-1, 2n-1).$$

$\Longrightarrow$ The inversion number of O is
$1 + \cdots + n - 1 = n(n-1)/2$.
$\Longrightarrow$ O is even iff $n \equiv 0$ or $1 \pmod 4$.

- I is obtained by permutating the two piles and then performing O. $\Longrightarrow$ I is even iff n and O have the same parity.
$\Longrightarrow$ I is even iff $n \equiv 0$ or $3 \pmod 4$.

- Thus $\langle O, I \rangle \leq \mathrm{Alt}(2n)$ iff $n \equiv 0 \pmod 4$.

$\langle O, I \rangle \longleftrightarrow$all the orderings by performing a sequence of Out-shuffles and In-shuffles

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Questions on $\langle O, I \rangle$

Question: Can $\langle O, I \rangle$ equal $\mathrm{Alt}(2n)$ when $n \equiv 0 \pmod 4$?

Question: Can $\langle O, I \rangle$ equal $\mathrm{Sym}(2n)$ when $n \not\equiv 0 \pmod 4$?

Answer: Both no.

Observation: out-shuffle and in-shuffle preserve the partition $\{0, 2n-1\}, \{1, 2n-2\}, \dots, \{n-1, n\}$.

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What is the goup structure of $\langle O, I \rangle$?

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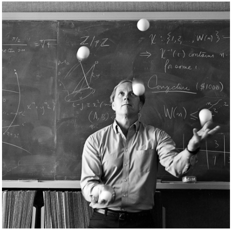
Diaconis-Graham-Kantor

Question: What is the goup structure of $\langle O, I \rangle$?

Answered by Diaconis, Graham and Kantor in 1983[1].

Persi Diaconis

ICM talk in 1990



Ron Graham

ICM talk in 1983

William M. Kantor

ICM talk in 1998

[1] P. Diaconis, R. L. Graham and W. M. Kantor, The mathematics of perfect shuffles., *Adv. Appl. Math.*, 4 (1983), 175–196.

Structures of $\langle O, I \rangle$

Size of each pile n	$\langle O, I \rangle$
$n \equiv 0 \pmod 4$, $n > 12$ and n is not a power of 2	$C_2^{n-1} \rtimes A_n$
$n \equiv 1 \pmod 4$	$C_2^n \rtimes A_n$
$n \equiv 2 \pmod 4$ and $n > 6$	$C_2 \wr S_n$
$n \equiv 3 \pmod 4$	$C_2^{n-1} \rtimes S_n$
$n = 2^f$ for some positive integer f	$C_2 \wr C_{f+1}$
$n = 6$	$C_2^6 \rtimes \mathrm{PGL}(2,5)$
$n = 12$	$C_2^{11} \rtimes M_{12}$

Table: Classification of $\langle O, I \rangle$

C_2^n: direct product of n copies of cyclic groups of order 2.
M_{12}: the sporadic Mathieu group on 12 points.

[1] P. Diaconis, R. L. Graham and W. M. Kantor, The mathematics of perfect shuffles., *Adv. Appl. Math.*, 4 (1983), 175–196.

A deck of kn cards with $k \geq 2$

- cut into k piles and then perfectly interleave them ($k!$ ways).

$$\begin{array}{c} 0 \\ 1 \\ \vdots \\ kn-1 \end{array} \longrightarrow \begin{array}{cccc} 0 & n & \cdots & (k-1)n \\ 1 & 1+n & \cdots & 1+(k-1)n \\ \vdots & \vdots & & \vdots \\ n-1 & 2n-1 & \cdots & kn-1 \end{array}$$

- Standard shuffle σ: $(i+jn)^\sigma = ik+j$.
- ρ_τ: $(i+jn)^{\rho_\tau} = i + j^\tau n$.
- The shuffle group on kn cards, denoted by $G_{k,kn}$, is generated by all possible shuffles $\rho_\tau \sigma$ for $\tau \in \mathrm{Sym}(\{0,\ldots,k-1\})$.
  $G_{k,kn} = \langle \rho_\tau \sigma \mid \tau \in \mathrm{Sym}(k) \rangle = \langle \rho_\tau, \sigma \mid \tau \in \mathrm{Sym}(k) \rangle$.

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Literature on $G_{k,kn}$ for $k \geq 3$

- Medvedoff and Morrison[2] in 1987 conjectured:
  - $G_{3,3n} \geq \mathrm{Alt}(3n)$ if n is not a power of 3;
  - $G_{4,4n} \geq \mathrm{Alt}(4n)$ if n is not a power of 2;
  - $G_{4,2^m} = \mathrm{AGL}(m,2) = C_2^m \rtimes \mathrm{GL}(m,2)$ if $m \geq 3$ is odd.

- In [2] they also proved:
  - $G_{k,kn} \leq \mathrm{Alt}(kn)$ if and only if either $n \equiv 0 \pmod 4$, or $n \equiv 2 \pmod 4$ and $k \equiv 0$ or $1 \pmod 4$.
  - $G_{k,k^m} = \mathrm{Sym}(k) \wr C_m$.

[2] S. Medvedoff and K. Morrison, Groups of perfect shuffles, *Math. Mag.*, 60 (1987), 3–14.

Literature on $G_{k,kn}$ for $k \geq 3$

- Cohen, Harmse, Morrison and Wright[3] confirmed the latter part of MM's conjecture when $k = 4$.

  ($G_{4,2^m} = \mathrm{AGL}(m, 2)$ for some odd integer $m \geq 3$)

- In [3] they also posed:

Shuffle Group Conjecture (2005)

For $k \geq 3$, if n is not a power of k and $(k, n) \neq (4, 2^f)$ for any positive integer f, then $G_{k,kn} \geq A_{kn}$.

[3] A. Cohen, A. Harmse, K.E. Morrison and S. Wright, Perfect shuffles and affine groups, 2005, https://aimath.org/morrison/Research/shuffles.

Literature on $G_{k,kn}$ for $k \geq 3$

- Amarra, Morgan and Praeger[4] confirmed the Shuffle Group Conjecture in the following cases:
  - $k > n$;
  - k and n are powers of the same integer $\ell \geq 2$;
  - k is a power of 2.
- They also opened up the study of "generalized shuffle groups".

[4] C. Amarra, L. Morgan and C. Praeger, Generalised shuffle groups, *Israel J. Math.*, 244 (2021), 807–856.

Our contribution

We confirmed Shuffle Group Conjecture for all the left cases.

Theorem

For $k \geq 3$, if n is not a power of k and $(k, n) \neq (4, 2^f)$ for any positive integer f, then $G_{k,kn} \geq A_{kn}$.

This thoerem leads to a complete classification of shuffle groups.

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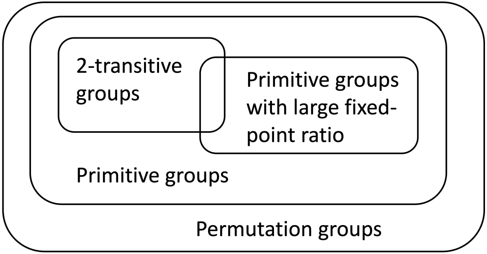
Outline of the proof

Prove: $G_{k,kn} = A_{kn}$ or S_{kn}, where $k \geqslant 3$, n is not a power of k and $(k, n) \neq (4, 2^f)$ for any positive integer f.

- $G_{k,kn}$ is 2-transitive.
- $G_{k,kn}$ has an element with large fixed-point ratio.
- classification of 2-transitive groups and primitive groups with large fixed-point ratio.
- Exclude all the candidates except from A_{kn} and S_{kn}.



2-transitivity

- $G_{k,kn}$ is 2-transitive iff its stabilizer on the point $kn-1$, denoted by H, is transitive on $\{0,\dots,kn-2\}$.
- In the proof of $G_{k,k^m} = S_k \wr C_m$, they found patterns by writing numbers $\{0,\dots,kn-2\}$ in base k.
- Let $n = k^s t$, where $k \nmid t$ and $t > 1$. Write
$$x = (x_s k^s + \cdots + x_1 k + x_0)t + X.$$
We have a bijection
$$x \longleftrightarrow (x_s,\dots,x_1,x_0;X).$$
- Find an inductive index $T(x) = |\{i \mid x_i = k-1\}|$.
- If $T(x) = 0$ then $x \in 0^H$. If $T(x) > 0$, then there eixsts $y \in x^H$ such that $T(y) < T(x)$.